1. (20 pts.) Polynomials
   (a) You are given three points, (0, 2), (1, 7) and (2, 1). What is the probability that a degree 5 polynomial in $GF_{11}$, with coefficients chosen uniformly at random, passes through these three points?

   (b) Given the same three points, what is the probability that a degree 1 polynomial in $GF_{11}$, with coefficients chosen uniformly at random, passes through these three points?
2. (20 pts.)  Diagonalization/Undecidability
Choose one of the following two questions.

- Use diagonalization to show that the set of functions from natural numbers to \( \{0, 1\} \) is uncountable.
  
  (Note: you must use diagonalization for this problem, not some other proof technique.)

- Suppose your professor asks you to write a “perfect compiler” that takes a C program as input and produces the shortest possible piece of assembly code (measured in number of bytes) with the same input-output specification as the original C program. Prove to him why this is not possible.
  
  (Hint: You may assume that the EQUIVALENCE problem from homework 10 is undecidable. In this problem you are given two pieces of code and asked to decide if they have the same output on every input.)
3. (20 pts.) Induction

A biased coin with \( \Pr[\text{heads}] = p \) and \( \Pr[\text{tails}] = q \) is tossed \( N \) times (\( N > 0 \)). Let the probability of an even number of heads after \( N \) tosses be \( A_N \) and the probability of an odd number of heads after \( N \) tosses be \( B_N \).

Show by induction that

\[
B_N - A_N = (p - q)^N
\]

- **Base case:**

- **Inductive hypothesis:**

- **Inductive step:**
4. (10 pts.) A Combinatorial Proof

Give a combinatorial proof of the identity

\[
\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}
\]

Do not give an algebraic proof using the definition of \( \binom{n}{k} \).

5. (10 pts.) Bayes’ Rule

Suppose \( \frac{1}{5} \) of all men are colorblind and \( \frac{1}{3} \) of all women are colorblind. Assume we pick a person uniformly at random from a room with 25 men and 75 women and the person turns out to be colorblind. What is the probability the colorblind person is male?
6. (20 pts.) Balls and Bins
Suppose $n$ indistinguishable balls are thrown into $n$ bins independently and uniformly at random.

(a) What is the probability of a particular bin containing exactly one ball?

(b) Compute the expected number of bins with exactly one ball.
    (Hint: Linearity of Expectation).
7. Extra Credit

This extra credit question will not be graded in terms of points. We will consider it only when assigning letter grades. You are strongly advised to only work on it when you are finished with the rest of the midterm.

Consider the following game: The dealer shuffles a regular deck of 52 cards and successively turns over one card at a time. After any card, you are allowed to say “stop”. If the next card is red, you win, and if it is black, you lose. You must say stop at some point during the game.

What is your optimal strategy (i.e. when should you say “stop”) and what is your probability of winning under this strategy?

(Hint: the answer is simple)