MATH 54 MIDTERM 1

Sep 23 2014 12:40-2:00pm

Section Number Section Leader	SOLUTIONS

Your Name	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. No material other than simple writing utensils may be used.

Your grade is determined from the highest scores on 4 of the following 5 problems.
So rather than working through everything, make sure your answers are careful and correct.

linear systems	1	
matrix algebra and inverse	2	
linear combinations and dependence	3	
abstract matrices and span	4	
elementary matrices and determinants	5	

[3] **1. (a)** Express the following matrix equation as linear system for variables x_i.

$$\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$7x_1 + 3x_2 = -5$$
$$-6x_1 - 3x_2 = 3$$

[3] **(b)** State what it means for $A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$ to have inverse $B = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}$.

(Make no calculations here – just algebraic statements.)

$$AB = I = BA$$

[4] **1. (c)** Demonstrate how to use a property of the inverse from (b) to find the solution to (a).

$\underline{x} = B\,\underline{b}$ solves $A\underline{x} = \underline{b}$

because $\underbrace{AB}_{=I \text{ by (b)}}\underline{b} = I\underline{b} = \underline{b}$

$$\Rightarrow \quad \underline{x} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -5+3 \\ 10-7 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}$$

[4] **1. (d)** For the matrix A from (b), use the facts $A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix}$ and $A\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ to give a solution of $A\mathbf{x} = \begin{bmatrix}0\\3\end{bmatrix}$ by superposition (~~without~~ i.e. using algebraic properties of matrix-vector multiplication rather than explicitly solving or computing a product).

$$\begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix} + 4\begin{bmatrix}1\\0\end{bmatrix} = A\begin{bmatrix}-1\\1\end{bmatrix} + 4A\begin{bmatrix}1\\-2\end{bmatrix} = A\left(\begin{bmatrix}-1\\1\end{bmatrix} + 4\begin{bmatrix}1\\-2\end{bmatrix}\right)$$

$$= A\begin{bmatrix}3\\-7\end{bmatrix}$$

$$\Rightarrow \boxed{\underline{x} = \begin{bmatrix}3\\-7\end{bmatrix}}$$

[6] **1. (e)** Describe the solutions of the following system in parametric vector form.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 3 \\ -2x_1 - 2x_2 - 4x_3 &= -8 \end{aligned}$$

$$\Updownarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ -2 & -2 & -4 & -8 \end{array}\right]$$

$$\Updownarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (-\text{row II}) \\ \\ (+2\,\text{row I}) \end{array}$$

$$\Updownarrow$$

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 3 \end{aligned}$$

$$\Updownarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 3 - x_3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

[8] **2. (a)** Compute or explain why the following expressions are undefined for $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$.

$3A$ $\qquad\qquad$ AA^T $\qquad\qquad$ $A^TA - AA^T$

$$3A = \begin{bmatrix} 6 & 0 & -3 \\ 0 & 9 & 3 \end{bmatrix}$$

$$\underbrace{A^TA}_{3\times 3} - \underbrace{AA^T}_{2\times 2} = \text{undefined}$$

matrices can only be added if # columns and # rows match

$$A^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2^2+(-1)^2 & -1\cdot 1 \\ 1\cdot -1 & 3^2+1^2 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix} = AA^T}$$

[6] **2. (b)** Calculate the inverse of $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$ by row reduction.

I

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & 0 & 1 & -3/5 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{=I} & & & 1 & -2 & 0 \\ & & & 0 & 1/2 & -3/10 \\ & & & 0 & 0 & 1/5 \end{array}\right]$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1/2 & -3/10 \\ 0 & 0 & 1/5 \end{bmatrix}}$$

and use it to calculate/ Give a

[6] **2. (c)** Check the $(1, 2)$ entry of ~~your~~ the result in (b). ~~using the~~ formula for A^{-1} in terms of cofactors.
(Hint: This entry is $\neq 0$.)

$$A^{-1} = \frac{1}{\det A}\left[C_{ij} = (-1)^{i+j} \det A_{ij}\right]^T$$

$$\det A = \text{product of diagonal since triangular} = 1 \cdot 2 \cdot 5 = 10$$

$$\Rightarrow (A^{-1})_{12} = \frac{1}{10}(-1)^{1+2} \underbrace{\det \begin{bmatrix} \cancel{1} & 4 & 6 \\ \cancel{0} & \cancel{2} & \cancel{3} \\ \cancel{0} & 0 & 5 \end{bmatrix}}_{4 \cdot 5 - 6 \cdot 0} = -\frac{20}{10} = \boxed{-2}$$

[6] **3. (a)** State a criterion and use it to decide whether the vectors $\begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ span $\mathbb{R}^3$.

n vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ if the corresponding square matrix has a pivot in each row $\Leftrightarrow$ reduced echelon form I $\Leftrightarrow$ is invertible

$\Updownarrow$

$\det \neq 0$

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 3 & -3 & 0 \\ -7 & 7 & -2 \end{bmatrix} = 1 \cdot (-3) \cdot (-2) = 6 \neq 0 \Rightarrow \boxed{\text{they span}}$$

lower triangular

---

alternative method

The vectors span $\mathbb{R}^n$ if the corresponding matrix has a pivot in each row

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -3 & 0 \\ -7 & 7 & -2 \end{bmatrix} \xrightarrow{\text{row replacements}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{-3} & 0 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$ has pivot in each row

$\Rightarrow$ column vectors $\boxed{\text{span } \mathbb{R}^3}$

[5] **3. (b)** Use your work in (a) and no further calculation to also decide and explain whether the vectors are linearly dependent.
(If you didn't solve (a), state a criterion for linear dependence and make the calculation here.)

n vectors in $\mathbb{R}^n$ are linearly dependent if the corresponding square matrix has a free variable

$\Leftrightarrow$ a column without pivot

$\Leftrightarrow$ reduced echelon form $\neq I$

$\underset{\text{as in (a)}}{\Leftrightarrow}$ vectors don't span $\mathbb{R}^n$

FALSE by (a) $\Longrightarrow$ $\boxed{\text{vectors are \underline{not} linearly dependent}}$

$\boxed{\text{alternative method}}$

The vectors are linearly dependent if the corresponding matrix has a free variable, i.e. column without pivot.

Echelon form from a): $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{-3} & 0 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$ has no free variable $\Rightarrow$ $\boxed{\text{vectors are not lin. dep.}}$

[3] **3. (c)** Use the fact that $\begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & -4 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$ to write $\mathbf{w} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$ as linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$.

$$3\underline{v}_1 + 1\underline{v}_2 + 0\underline{v}_3 = \underline{w}$$

[6] **3. (d)** Decide and explain whether there are weights other than the ones found in (c) that allow to write $\mathbf{w}$ as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

The weights are solutions of $\underbrace{[\underline{v}_1\ \underline{v}_2\ \underline{v}_3]}_{V}\underline{x} = \underline{w}$.

Since V is square, solutions are unique exactly when $\underline{V \text{ is invertible}}$.

$\Updownarrow$

$\det V \neq 0 \longrightarrow$ $\boxed{\text{weights not unique}}$

$$\det\begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & -4 \\ 3 & 0 & -6 \end{bmatrix} = -(-2)\det\begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} = 2(-12-(-12)) = \underline{\underline{0}}$$

expand by 2^{nd} column

<u>alternative method</u>:

The weights are solutions of $[\underline{v}_1\ \underline{v}_2\ \underline{v}_3]\underline{x} = \underline{w}$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 2 & 0 & -4 & 6 \\ 3 & 0 & -6 & 9 \end{array}\right] \xrightarrow{\text{row op.}} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 4 & 1 \\ 0 & \boxed{4} & -8 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

no pivot

consistent but free variable

$\Rightarrow$ sol^{ns} = weights $\underline{x}$ $\boxed{\text{not unique}}$

4. Give counterexamples or justify the following statements **just using definitions and algebra (no theorems)**.

[5] **(a)** Suppose A is an $m \times n$ matrix and there exists a matrix D so that $AD = I$. Then the columns of A span $\mathbb{R}^m$.

Solutions of $A\underline{x} = \underline{b}$ always exist for $\underline{b}$ in $\mathbb{R}^m$

since $\underline{x} = D\underline{b}$ solves $A\underline{x} = AD\underline{b} = I\underline{b} = \underline{b}$.

So any $\underline{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A,

i.e. the columns span $\mathbb{R}^m$.

[5] **(b)** Suppose A is an $m \times n$ matrix and there exists a matrix D so that $AD = I$. Then solutions to $A\mathbf{x} = \mathbf{b}$ are unique.

$\boxed{\text{FALSE}}$

$$\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{D} = 1 = I_1$$

but $A\underline{x} = \underline{0}$ has solutions $\underline{x} = \begin{bmatrix} 0 \\ c \end{bmatrix}$

for any scalar c

4.continued

[5] **(c)** Can $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contain vectors that are not in $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$?
(Give an example or reasoning.)

$\boxed{\text{NO}}$ If $\underline{w}$ in span$\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$ then for some c_1, c_2, c_3

$$\underline{w} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$$
$$= c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + 0\underline{v}_4$$

which is a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$,
so $\underline{w}$ is in that span, too.

[5] **(d)** Explain What ~~would~~ equality of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ would imply about linear (in)dependence of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

It means $\underline{v}_4$ lies in $\operatorname{span}\{\underline{v}_1 .. \underline{v}_4\}$ and hence in $\operatorname{span}\{\underline{v}_1 .. \underline{v}_3\}$, which implies $\boxed{\text{linear dependence}}$ of $\underline{v}_1 \dots \underline{v}_4$.

[4] **5. (a)** Write down the elementary 3×3 matrices that represent the following row operations:

- adding six times the second row to the first row: $E_1 = \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- scaling the first row by $\frac{1}{3}$: $E_2 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- interchanging the first and third row : $E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

[4] **(b)** With the matrices E_1, E_2, E_3 from (a) and $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 15 & 0 & 0 \end{bmatrix}$ calculate $E_1E_2E_3A$.

interchange

$$\downarrow$$

$$\begin{bmatrix} 15 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = E_3A$$

scale

$$\downarrow$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = E_2E_3A$$

replace

$$\downarrow$$

$$\boxed{\begin{bmatrix} 5 & -6 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = E_1E_2E_3A}$$

[6] **5. (c)** With the matrices E_1, E_2, E_3 from (a), give and explain simple formulas that relate the following for any 3×3 matrix $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$.

- V_{123} = the volume of the parallelepiped determined by the column vectors of $E_1E_2E_3B$
- V_{321} = the volume of the parallelepiped determined by the column vectors of $E_3E_2E_1B$
- V = the volume of the parallelepiped determined the vectors $E_3\mathbf{b}_1, E_3\mathbf{b}_2, E_3\mathbf{b}_3$

$$V_{123} = |\det(E_1E_2E_3B)| = \overset{1}{\overset{\shortparallel}{|\det E_1|}} \cdot \overset{1/3}{\overset{\shortparallel}{|\det E_2|}} \cdot \overset{-1}{\overset{\shortparallel}{|\det E_3|}} \cdot |\det B|$$

$$V_{321} = |\det(E_3E_2E_1B)| = |\det E_3| \cdot |\det E_2| \cdot |\det E_1| \cdot |\det B|$$

$$V = |\det(E_1B)| = |\det E_1| \cdot |\det B|$$

$$\Rightarrow \boxed{V_{123} = V_{321} = \frac{1}{3}V}$$

[6] **5. (d)** Let A, B, C be 4×4 matrices with $\det A = 2$, $\det B = -1$, $\det C = 5$. Compute

$\det(BC^{-1}A) =$ $\det B \cdot \frac{1}{\det C} \cdot \det A = (-1) \cdot \frac{1}{5} \cdot 2 = \boxed{\frac{-2}{5}}$

$\det(2B) =$ $2^n \det B \underset{(n=4)}{=} \boxed{-16}$

$\det(C^T A) - \det(A^T C) =$ $\underset{\det C}{\underset{\parallel}{\det C^T}} \cdot \det A - \underset{\det A}{\underset{\parallel}{\det A^T}} \cdot \det C = \boxed{0}$