MATH 54 MIDTERM 1

Sep 23 2014 12:40-2:00pm

Section Number	
Section Leader	

Your Name	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. No material other than simple writing utensils may be used. *In the event of an emergency or fire alarm leave your exam (closed) on your seat and meet with your GSI outside.*
If you need to use the restroom, leave your exam with your GSI while out of the room.

Your grade is determined from the highest scores on 4 of the following 5 problems.

So rather than working through everything, make sure your answers are careful and correct.

linear systems	1	
matrix algebra and inverse	2	
linear combinations and dependence	3	
abstract matrices and span	4	
elementary matrices and determinants	5	

[3] **1. (a)** Express the following matrix equation as linear system for variables x_i.

$$\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

[3] **(b)** State what it means for $A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$ to have inverse $B = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}$.

(Make no calculations here – just algebraic statements.)

[4] **1. (c)** Demonstrate how to use a property of the inverse from (b) to find the solution to (a).

[4] **1. (d)** For the matrix A from (b), use the facts $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to give a solution of $A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ by superposition (using algebraic properties of matrix-vector multiplication rather than explicitly solving or computing a product).

[6] **1. (e)** Describe the solutions of the following system in parametric vector form.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 3 \\ -2x_1 - 2x_2 - 4x_3 &= -8 \end{aligned}$$

[8] **2. (a)** Compute or explain why the following expressions are undefined for $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$.

$3A$ $\qquad\qquad$ AA^T $\qquad\qquad$ $A^TA - AA^T$

[6] **2. (b)** Calculate the inverse of $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$ by row reduction.

[6] **2. (c)** Give a formula for A^{-1} in terms of cofactors and use it to calculate/check the $(1, 2)$ entry of the result in (b). (Hint: This entry is $\neq 0$.)

[6] **3. (a)** State a criterion and use it to decide whether the vectors $\begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ span $\mathbb{R}^3$.

[5] **3. (b)** Use your work in (a) and no further calculation to also decide and explain whether the vectors are linearly dependent.
(If you didn't solve (a), state a criterion for linear dependence and make the calculation here.)

[3] **3. (c)** Use the fact that $\begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & -4 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$ to write $\mathbf{w} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$ as linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$.

[6] **3. (d)** Decide and explain whether there are weights other than the ones found in (c) that allow to write $\mathbf{w}$ as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

4. Give counterexamples or justify the following statements **just using definitions and algebra (no theorems)**.

[5] **(a)** Suppose A is an $m \times n$ matrix and there exists a matrix D so that $AD = I$. Then the columns of A span $\mathbb{R}^m$.

[5] **(b)** Suppose A is an $m \times n$ matrix and there exists a matrix D so that $AD = I$. Then solutions to $A\mathbf{x} = \mathbf{b}$ are unique.

4.continued

[5] **(c)** Can $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contain vectors that are not in $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$?
(Give an example or reasoning why it's impossible.)

[5] **(d)** Explain what equality of $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ would imply about linear (in)dependence of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

[4] **5. (a)** Write down the elementary 3×3 matrices that represent the following row operations:

- adding six times the second row to the first row: $E_1 =$

- scaling the first row by $\frac{1}{3}$: $E_2 =$

- interchanging the first and third row : $E_3 =$

[4] **(b)** With the matrices E_1, E_2, E_3 from (a) and $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 15 & 0 & 0 \end{bmatrix}$ calculate $E_1 E_2 E_3 A$.

[6] **5. (c)** With the matrices E_1, E_2, E_3 from (a), repeated below, give and explain simple formulas that relate the following for any 3×3 matrix $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$.

- $V_{123} =$ the volume of the parallelepiped determined by the column vectors of $E_1E_2E_3B$
- $V_{321} =$ the volume of the parallelepiped determined by the column vectors of $E_3E_2E_1B$
- $V =$ the volume of the parallelepiped determined the vectors $E_1\mathbf{b}_1, E_1\mathbf{b}_2, E_1\mathbf{b}_3$

For reference:

E_1 : adding six times the second row to the first row

E_2 : scaling the first row by $\frac{1}{3}$

E_3 : interchanging the first and third row

[6] **5. (d)** Let A, B, C be 4×4 matrices with $\det A = 2$, $\det B = -1$, $\det C = 5$. Compute

$\det(BC^{-1}A) =$

$\det(2B) =$

$\det(C^T A) - \det(A^T C) =$