Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 8 problems, each worth 10 points. We will grade all 8 problems, and count your top 6 scores.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total Possible	60	

Problem 1) True or False. Decide if each of the following statements is TRUE or FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer box of the chart. (Each correct answer receives 2 points, incorrect answers or blank answers receive 0 points.)

Statement	1	2	3	4	5
Answer	T	T	T	T	F

1) For any inner product on $\mathbb{R}^2$, if vectors $\mathbf{u}, \mathbf{v}$ satisfy $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1$ and $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = \|\mathbf{u}, \mathbf{u}\|^2 + \|\mathbf{v}, \mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle$ so $2 = 1+1-2\langle \mathbf{u}, \mathbf{v}\rangle$ so $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

2) In the vector space of continuous functions on the interval $[-1, 1]$ with inner product

$$\langle f(t), g(t)\rangle = \int_{-1}^{1} f(t)g(t)\, dt$$

the functions $\cos(t)$ and $\sin(t)$ are orthogonal.

$\cos(t)$ is even and $\sin(t)$ is odd so $\cos(t)\sin(t)$ is odd so its integral over $[-1, 0]$ is the negative of its integral over $[0, 1]$.

3) If A is symmetric and U is orthogonal, then UAU^{-1} is symmetric.

A is symmetric so $A^T = A$ and U is orthogonal so $U^{-1} = U^T$, $(U^{-1})^T = (U^T)^T = U$ hence $(UAU^{-1})^T = (U^{-1})^T A^T U^T = UAU^{-1}$.

4) If a 2×2 matrix A has eigenvalues λ_1, λ_2, then its characteristic polynomial is equal to

$$\chi_A(t) = t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2$$

If A has an eigenvalue λ_1, then it is similar to an upper triangular matrix $\begin{bmatrix}\lambda_1 & u \\ 0 & \lambda_2\end{bmatrix}$ so $\chi_A(t) = \det \begin{bmatrix}\lambda_1 - t & u \\ 0 & \lambda_2 - t\end{bmatrix} = t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2$.

5) Let V be the vector space of differentiable functions on the real line. The linear transformation $T : V \to V$ given by $T(y) = y'' - e^{-t}y' + 2y$ is injective.

We can always find a nonzero y so that $T(y) = 0$ (in fact T has a 2-dimensional null space) so T is not injective.

Problem 2) Multiple Choice. There is a single correct answer to each of the following questions. Determine what it is and write the letter in the answer box of the chart. You do not need to justify your answers. (Each correct answer receives 2 points, incorrect answers or blank answers receive 0 points.)

Question	1	2	3	4	5
Answer	B	B	C	B	A

1) Which of the following matrices is similar to $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$?

A) $\begin{bmatrix} -4 & 1 \\ 0 & 5 \end{bmatrix}$ *B)* $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ *C)* $\begin{bmatrix} 5 & 1 \\ 0 & -4 \end{bmatrix}$ *D)* $\begin{bmatrix} 1 & 6 \\ 0 & -2 \end{bmatrix}$ *E)* none of the preceding.

The matrix's eigenvalues are $2, -1$ so it is similar to any upper triangular matrix with those diagonal entries.

2) For some basis B of the vector space $\mathbb{R}^2$, the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have coordinates $[\mathbf{u}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, [\mathbf{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is the vector $\mathbf{w}$ with coordinates $[\mathbf{w}]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

A) $\begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$ *B)* $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$ *C)* $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ *D)* $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$ *E)* not determined by the data.

$P_B^{-1} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ so $P_B = \frac{1}{4}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{w} = P_B[\mathbf{w}]_B$.

3) For which pair of real numbers (a, b) is the matrix $\begin{bmatrix} 1 & -2 & -1 \\ -1 & a & 1 \\ 3 & -6 & b \end{bmatrix}$ rank one?

$A)$ $(-1, -3)$ $B)$ $(2, -1)$ $C)$ $(2, -3)$ $D)$ $(-2, 3)$ $E)$ none of the preceding.

Need each row to be a scale of the first.

4) What is the sum of the dimensions of the null space and column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}?$$

$A)$ 4 $B)$ 5 $C)$ 6 $D)$ 7 $E)$ 8

Dimension of domain = rank + dimension of null space.

5) For which triples of real numbers (a, b, c) does the linear system

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a & 0 & -1 \\ 1 & b & c \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

have a solution for any $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$?

$A)$ $(0, 1, 2)$ $B)$ $(2, 1, 0)$ $C)$ $(2, 2, 1)$ $D)$ $(1, 0, 2)$ $E)$ none of the preceding.

The matrix's determinant is $b(2-a)$ so is nonzero and hence invertible when $a \neq 2, b \neq 0$.

Problem 3) 1) (5 points) Find the orthogonal projection of the vector $\mathbf{b}$ to the subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}, \mathbf{v}$ where

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Use Gram-Schmidt to find orthogonal basis consisting of $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and

$$\mathbf{w} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Then the orthogonal projection is

$$\frac{\langle \mathbf{b}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u} + \frac{\langle \mathbf{b}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$

2) (5 points) Find a least-squares approximate solution to the equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Using the first part of the problem, we must solve

$$\begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first two equations then require $x = 3/4$, $y = 1/4$.

Problem 4) 1) (5 points) Find the general solution of the second order ODE

$$y'' - 2y' - 3y = 0$$

Auxiliary equation is $r^2 - 2r - 3 = (r-3)(r+1) = 0$ with roots $3, -1$ so general solution is of the form

$$y = c_1 e^{3t} + c_2 e^{-t}$$

2) (5 points) Find the general solution of the second order ODE

$$y'' - 2y' - 3y = 10\cos(t)$$

Try $y = a\cos(t) + b\sin(t)$.

Then $y'' - 2y' - 3y = (-a\cos(t) - b\sin(t)) - 2(-a\sin(t) + b\cos(t)) - 3(a\cos(t) + b\sin(t)) = (-4a - 2b)\cos(t) + (2a - 4b)\sin(t)$.

So we must solve the linear system

$$\begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

The second equation implies $a = 2b$ and then the first equation implies $a = -2, b = -1$.

Using the first part of the problem, the general solution is thus

$$y = -2\cos(t) - \sin(t) + c_1 e^{3t} + c_2 e^{-t}$$

Problem 5) 1) (5 points) Find a basis of solutions of the equation

$$\mathbf{y}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}(t)$$

Characteristic polynomial is $\chi_A(t) = t^2 - 2t - 3$ so eigenvalues are $-1, 3$. Corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus basis of solutions is

$$e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2) (5 points) Write down a 3×3 matrix A such that the equation $\mathbf{y}'(t) = A\mathbf{y}(t)$ has a basis of solutions

$$\mathbf{y}_1(t) = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_3(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Eigenvalues are $-1, 2, 0$ with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Problem 6) (10 points) Use separation of variables to find a solution $u = u(x,t)$ of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u$$

satisfying $u(x,0) = e^x$ and $u(x,t) \to 0$ as $t \to \infty$ for all x.

Set $u(x,t) = X(x)T(t)$.
Then

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t) \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

Thus $X(x)T''(t) = X''(x)T(t) + X(x)T(t)$ so

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + 1$$

The left hand side is independent of x and the right hand side is independent of t so they both must be equal to some constant

$$\frac{T''(t)}{T(t)} = c = \frac{X''(x)}{X(x)} + 1$$

Thus we have the ODEs

$$T''(t) - cT(t) = 0 \qquad X''(x) - (c-1)X(x) = 0$$

with bases of solutions

$$e^{(\sqrt{c})t}, e^{-(\sqrt{c})t} \qquad e^{(\sqrt{c-1})x}, e^{(-\sqrt{c-1})x}$$

The initial condition requires we take $c = 2$ and the solution e^x.
The limit condition then requires we take a solution $ae^{-(\sqrt{2})t}$.
Thus altogether we find $ae^x e^{-(\sqrt{2})t}$ and the initial condition requires we take $a = 1$ to conclude $y = e^x e^{-(\sqrt{2})t}$.

Problem 7) Consider the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for the function $|x|$ on the interval $[-\pi, \pi]$.

1) (5 points) Calculate the coefficients a_n, for all n.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx)\ dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx)\ dx$$

If $n = 0$, we find

$$a_0 = \pi$$

If $n > 0$, using $\frac{d}{dx}(x \sin(nx)) = \sin(nx) + nx \cos(nx)$, we integrate by parts to find

$$\frac{2}{\pi} \int_0^{\pi} x \cos(nx)\ dx = \frac{2}{n\pi}([x \sin(nx)]|_0^{\pi} - [-\cos(nx)/n]|_0^{\pi}) = \frac{2}{n^2\pi} \cos(nx)|_0^{\pi}$$

If n is even and $n > 0$, we find

$$a_n = \frac{-4}{n^2\pi}$$

If n is odd, we find

$$a_n = 0$$

2) (5 points) Calculate the coefficients b_n, for all n.

$b_n = 0$, for all n, since $|x|$ is even.

Problem 8) The following assertions are FALSE. Provide a counterexample (4 points each) along with a clear and brief justification no longer than one sentence (1 point each).

1) (5 points) If A is a 2×2 symmetric matrix with positive integer entries, then any eigenvalue of A is positive or zero.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$\chi_A(t) = t^2 - 2t - 3$ so eigenvalues are $-1, 3$.

2) (5 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is an injective linear transformation. For any given basis of $\mathbb{R}^3$, there is a basis of $\mathbb{R}^2$ such that the matrix of T takes the form

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Take T to be the linear transformation

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}$$

with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Then we can not choose an alternative basis of $\mathbb{R}^2$ so that $[T]$ takes the desired form since

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is not in the image of T so can not be in the column span of $[T]$.