Name (Last, First): ____________________

Student ID: ____________________

Circle your section:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 201 | Shin | 8am | 71 Evans | 212 | Lim | 1pm | 3105 Etcheverry |
| 202 | Cho | 8am | 75 Evans | 213 | Tanzer | 2pm | 35 Evans |
| 203 | Shin | 9am | 105 Latimer | 214 | Moody | 2pm | 81 Evans |
| 204 | Cho | 9am | 254 Sutardja Dai | 215 | Tanzer | 3pm | 206 Wheeler |
| 205 | Zhou | 10am | 254 Sutardja Dai | 216 | Moody | 3pm | 61 Evans |
| 206 | Theerakarn | 10am | 179 Stanley | 217 | Lim | 8am | 310 Hearst |
| 207 | Theerakarn | 11am | 179 Stanley | 218 | Moody | 5pm | 71 Evans |
| 208 | Zhou | 11am | 254 Sutardja Dai | 219 | Lee | 5pm | 3111 Etcheverry |
| 209 | Wong | 12pm | 3 Evans | 220 | Williams | 12pm | 289 Cory |
| 210 | Tabrizian | 12pm | 9 Evans | 221 | Williams | 3pm | 140 Barrows |
| 211 | Wong | 1pm | 254 Sutardja Dai | 222 | Williams | 2pm | 220 Wheeler |

If none of the above, please explain: ____________________

This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points. We will grade all 6 problems, and count your top 5 scores.

| Problem | Maximum Score | Your Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| Total Possible | 50 | |

**Problem 1)** Decide if the following statements are ALWAYS TRUE or SOMETIMES FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer boxes of the chart. (Correct answers receive 2 points, incorrect answers or blank answers receive 0 points.)

| Statement | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Answer | TRUE | FALSE | FALSE | FALSE | TRUE |

1) The matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar.

2) The matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ are similar.

3) For $2 \times 2$ matrices $A$ and $B$, if $\mathbf{v}$ is an eigenvector of $AB$, then $B\mathbf{v}$ is an eigenvector of $A$.

4) If a $3 \times 3$ matrix $A$ is diagonalizable with eigenvalues $\pm 1$, then it is an orthogonal matrix.

5) If $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

**Problem 2)** Indicate with an **X** in the chart all of the answers that satisfy the questions below. You do not need to justify your answers. It is possible that any number of the answers satisfy the questions. (A completely correct row of the chart receives 2 points, a partially correct row receives 1 point, but any incorrect X in a row leads to 0 points.)

| | $(a)$ | $(b)$ | $(c)$ | $(d)$ | $(e)$ |
|---|---|---|---|---|---|
| Question 1 | | X | | X | X |
| Question 2 | X | | X | | |
| Question 3 | | X | | X | X |
| Question 4 | | X | | X | |
| Question 5 | X | X | | X | X |

1) Which of the following conditions guarantees an $n \times n$ real matrix $A$ is diagonalizable with real eigenvalues?

$a$) Every eigenvalue of $A$ has an eigenvector.
$b$) There is a basis of $\mathbb{R}^n$ consisting of real eigenvectors for $A$.
$c$) $\det(A - \lambda I_n) = \lambda^n - \lambda^{n-1}$ and $\dim Nul(A) = 1$.
$d$) $\det(A - \lambda I_n) = \lambda^n - \lambda^{n-2}$ and $\dim Nul(A) = n - 2$.
$e$) The inverse of $A$ is diagonalizable with real eigenvalues.

2) For what $h$ is the matrix

$$\begin{bmatrix} 1 & -h^2 & 2h \\ 0 & 2h & h \\ 0 & 0 & h^2 \end{bmatrix}$$

diagonalizable with real eigenvalues?

$a$) $h = -2$ $b$) $h = -1$ $c$) $h = 0$ $d$) $h = 1$ $e$) $h = 2$

3) Which of the following linear transformations $T : P_2 \to P_2$ have rank 1?

$$a)\ T(p(x)) = p'(x) \quad b)\ T(p(x)) = p''(x) \quad c)\ T(p(x)) = (1+x)p'(x)$$

$$d)\ T(p(x)) = (1+x)p''(x) \quad e)\ T(p(x)) = (1+x)p(1)$$

4) Which of the following are a basis $B$ of $\mathbb{R}^3$ so that for $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ we have $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$?

$$a) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad b) \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \quad c) \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 9 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e) \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5) Which of the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}$ are given by an orthogonal matrix $A$?

*a*) Reflection across the line $x = y$.
*b*) Rotation by $\pi/4$ about the origin.
*c*) A shear transformation fixing the line $y = 0$.
*d*) Reflection across the line $x = y$ followed by reflection across the line $x = 0$.
*e*) Scaling by 2 followed by rotation by $\pi/4$ about the origin followed by scaling by $1/2$.

**Problem 3)** a) (4 points) Find the eigenvalues and a basis consisting of eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \\ -3 & -3 & 1 \end{bmatrix}$$

*Solution*: $\det(A - \lambda I) = -\lambda(1-\lambda)(-2-\lambda)$ so eigenvalues are $0, 1, -2$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

b) (3 points) Find the coordinates of the vector

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

with respect to the basis of eigenvectors.

*Solution*:

$$\mathbf{v} = 1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

so coordinates are

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

c) (3 points) Calculate $A^{2014}\mathbf{v}$.

*Solution*:

$$A^{2014}\mathbf{v} = A^{2014}(\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3) = -\mathbf{v}_2 + (-2)^{2014}\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 + (-2)^{2014} \\ -1 + (-2)^{2014} \end{bmatrix}$$

**Problem 4)** a) (4 points) Calculate the matrix $[T]$ of the linear transformation

$$T : P_2 \to \mathbb{R}^3 \qquad T(p(x)) = \begin{bmatrix} p(1) \\ p'(0) - p'(1) \\ p'(0) + p'(1) \end{bmatrix}$$

with respect to the basis $B = \{1, 1 + x, 1 + x + x^2\}$ of $P_2$ and the standard basis of $\mathbb{R}^3$.

*Solution*: Columns of $[T]$ result from applying $T$ to the basis vectors of $B$ and expanding in terms of the standard coordinate basis.

$$[T] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 2 & 4 \end{bmatrix}$$

b) (4 points) Find bases of $P_2$ and $\mathbb{R}^3$ such that the matrix of $T$ satisfies

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Solution*: Take for example the standard basis $1, x, x^2$ of $P_2$ and the resulting basis of images

$$T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T(x) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad T(x^2) = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

c) (2 points) Find bases of $P_2$ and $\mathbb{R}^3$ such that the matrix of $T^{-1}$ satisfies

$$[T^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Solution*: Take the same bases as in part b).

**Problem 5)** Consider the subspace $W$ of $\mathbb{R}^4$ spanned by

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$$

a) (4 points) Find a nonzero vector $\mathbf{w}$ in $W$ orthogonal to $\mathbf{u}$.

*Solution*: Set $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. We want $\mathbf{w} \cdot \mathbf{u} = 0$. We find $\mathbf{w} \cdot \mathbf{u} = 9a + 9b$. So for example take $a = 1, b = -1$ and so

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}$$

b) (3 points) Find the orthogonal projection of the vector

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

to the subspace $W$.

*Solution*: We calculate

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} + \frac{\mathbf{y} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} = \frac{1}{9}\mathbf{u} + \frac{-7}{9}\mathbf{w} = \begin{bmatrix} 1/9 \\ -7/9 \\ 12/9 \\ 16/9 \end{bmatrix}$$

c) (3 points) Find the orthogonal projection of the vector $\mathbf{y}$ to the orthogonal subspace $W^{\perp}$.

*Solution*: Take

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 3 - 1/9 \\ -1 - (-7/9) \\ 2 - 12/9 \\ 1 - 16/9 \end{bmatrix} = \begin{bmatrix} 26/9 \\ -2/9 \\ 6/9 \\ -7/9 \end{bmatrix}$$

**Problem 6)** (10 points) Fill in the blanks (each worth 1/2 a point) in the proof of the following assertion.

**Assertion.** If a $2 \times 2$ matrix $A$ satisfies $\det(A - \lambda I) = \lambda^2$, then $A^2 = 0$.

*Proof.* Since $\underline{\quad \det(A - \lambda I) = \lambda^2 \quad}$, the only $\underline{\quad \text{eigenvalue} \quad}$ of $A$ is 0.

There must be a corresponding $\underline{\quad \text{eigenvector} \quad}$, which we will call $\mathbf{v}$, because

$\underline{\quad \det(A) = 0 \quad}$ implies $\underline{\quad A \quad}$ is not invertible, and therefore

$\underline{\quad Nul(A) \quad}$ must be nontrivial.

Choose any $\mathbf{w}$ linearly independent from $\mathbf{v}$. Thus the pair $\mathbf{v}, \mathbf{w}$ is a $\underline{\quad \text{basis} \quad}$,

which we will call $B$, because $\mathbf{v}, \mathbf{w}$ must also $\underline{\quad \text{span } \mathbb{R}^2 \quad}$. Thus there exist $a, b$

so that $A\mathbf{w} = a\mathbf{v} + b\mathbf{w}$. The matrix of $A$ with respect to $B$ is then

$$[A]_B = \begin{bmatrix} \underline{\;0\;} & \underline{\;a\;} \\ \underline{\;0\;} & \underline{\;b\;} \end{bmatrix}$$

We see that $\underline{\quad b \quad} = 0$, since the $\underline{\quad \text{diagonal} \quad}$ entries

of any $\underline{\quad \text{triangular} \quad}$ matrix are its $\underline{\quad \text{eigenvalues} \quad}$.

Finally, let $P_B$ be the matrix with columns $\mathbf{v}, \mathbf{w}$. Then $A = \underline{\quad P_B [A]_B P_B^{-1} \quad}$.

Since it is easy to see that $\underline{\quad [A]_B^2 \quad} = 0$, we also find

$$A^2 = \underline{\quad (P_B [A]_B P_B^{-1})^2 \quad} = \underline{\quad P_B [A]_B^2 P_B^{-1} \quad} = 0.$$