ME 106 Midterm #1 Solution

Problem 1. Note: r is a variable $r = [R_1, R_2]$, $K = R_2/R_1$ (Kappa)

a) Let $r = R_2$, $u_\theta\big|_{R_2} = \frac{\omega}{K^2-1}\left[\frac{R_2^2}{R_2} - R_2\right] = 0$.

Let $r = R_1$, $u_\theta\big|_{R_1} = \frac{\omega}{k^2-1}\left[\frac{R_2^2}{R_1} - R_1\right]$

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$$= \frac{\omega}{k^2-1}[K^2-1]R_1$$

$$= \omega R_1$$

No-slip boundary condition is satisfied on the cylinder walls.

b) $\tau = \mu\left[r\frac{d}{dr}\left[\frac{u_\theta}{r}\right]\right]$

$$= \mu \cdot r\frac{d}{dr}\left(\frac{\omega}{k^2-1}\left(\frac{R_2^2}{r^2}-1\right)\right)$$

$$= \mu\frac{\omega r}{k^2-1}\frac{d}{dr}\left(\frac{R_2^2}{r^2}-1\right)$$

$$= \mu\frac{\omega r R_2^2}{k^2-1}\left(-\frac{2}{r^3}\right)$$

$$= \frac{2\mu\omega R_2^2}{(1-k^2)r^2}$$

10 On inner wall, $r = R_1$,

$$\tau_1 = \frac{2\mu\omega R_2^2}{(1-k^2)R_1^2} = 2\mu\omega\frac{k^2}{1-k^2}$$

10 On outer wall, $r = R_2$,

$$\tau_2 = -\frac{2\mu\omega R_2^2}{(1-K)^2 R_2^2} = -2\mu\omega\frac{1}{1-K^2}$$

$\boxed{\text{1 pt for opposite sign}}$

(c)

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$$\frac{\tau_2}{\tau_1} = \frac{-2\mu\omega \frac{1}{1-k^2}}{2\mu\omega \frac{k^2}{1-k^2}} = -\frac{1}{k^2} = -\frac{R_1^2}{R_2^2}$$

tau_1 > - tau_2

Total torque on fluid per unit length into the paper.

$$\sum M = M_1 + M_2$$

$$= (2\pi R_1 \tau_1) \cdot R_1 + (2\pi R_2 \tau_2) R_2$$

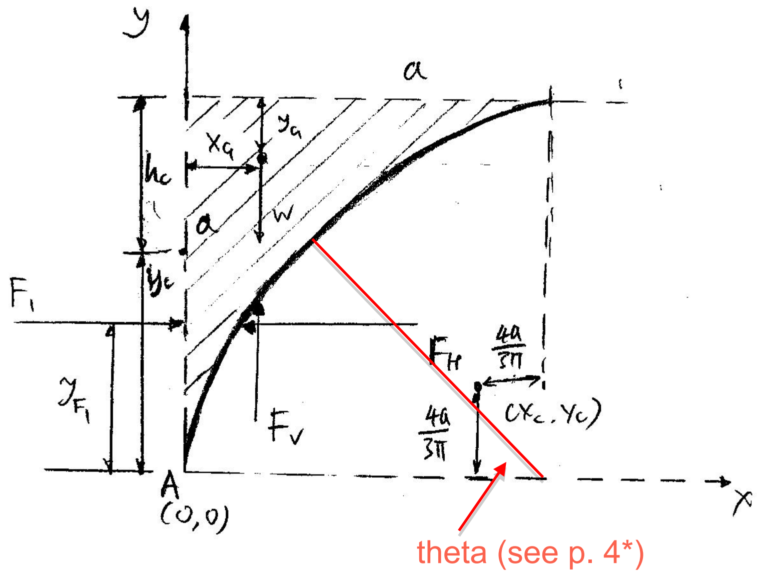
$$= 4\pi\mu\omega R_1^2 \frac{k^2}{1-k^2} - 4\pi\mu\omega R_2^2 \frac{1}{1-k^2}$$

$$= \frac{4\pi\mu\omega}{1-k^2}\left(R_1^2 k^2 - R_2^2\right)$$

2

$$= 0$$

Problem 2



* P_0 on the top surface and curved surface cancels.

* Consider force and moment balance on the shaded volume of fluid.

* All forces are per unit distance into the paper.

* Origin at A.

$\gamma = \rho_w g$

(a) $F_H = F_1$ on vertical surfaces. F_H and F_1 colinear.

For the vertical surface, $h_c = \frac{a}{2}$, $A = a$ (per unit distance into paper)

(10) $F_1 = \gamma h_c A = \gamma \frac{a}{2} \cdot a = \frac{1}{2} \gamma a^2$

Line of action of F_1 is along $y = y_{F_1}$.

$y_c = \frac{a}{2}$ and y_{F_1} is below y_c by $\frac{I_{zc}}{h_c A}$ — z-axis pointing out of paper.

(10) $y_{F_1} = y_c - \frac{I_{zc}}{h_c \cdot A} = \frac{a}{2} - \frac{\frac{1}{12} a^3}{\frac{1}{2} a \cdot a} = \frac{1}{3} a$

F_x and F_H has the same magnitude and line of action but opposite direction.

$F_x = \frac{1}{2} \gamma a^2 = \frac{1}{2} \rho_w g a^2$ in the positive x-direction with line of action along $y = \frac{1}{3} a$.

(b) $W = \gamma \left(a^2 - \frac{\pi a^2}{4} \right) = \gamma a^2 \left(1 - \frac{\pi}{4} \right)$ (10)

The centroid of the quarter circle is located at

$x_c = a - \frac{4a}{3\pi}$ $\qquad y_c = \frac{4a}{3\pi}$

To get x_a, consider area moment about y-axis,

$$x_a \cdot \left(a^2 - \frac{\pi a^2}{4} \right) + x_c \cdot \left(\frac{\pi a^2}{4} \right) = \underbrace{\frac{a}{2}}_{\text{centroid of square}} \cdot \underbrace{a^2}_{\text{area of square}}$$

$$x_a = \frac{\frac{1}{2}a^3 - \left(a - \frac{4a}{3\pi}\right)\frac{\pi a^2}{4}}{a^2 - \frac{\pi a^2}{4}}$$

(10) $$= \frac{\frac{1}{2} - \frac{\pi}{4} + \frac{1}{3}}{1 - \frac{\pi}{4}}\, a = \frac{\frac{5}{6} - \frac{\pi}{4}}{1 - \frac{\pi}{4}}\, a$$

$$= \frac{10 - 3\pi}{12 - 3\pi}\, a \cong 0.22a$$

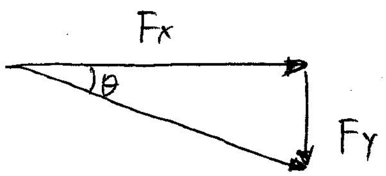
$F_v = W$ and F_v and W are colinear.

F_y and F_v have the same magnitude and line of action but opposite direction.

$F_y = \gamma a^2 (1 - \frac{\pi}{4})$. in the negative y-direction along

$x = x_a = \frac{10 - 3\pi}{12 - 3\pi}\, a. \cong 0.22a$

(c)



$$\tan\theta = \frac{F_y}{F_x} = \frac{\gamma a^2 (1 - \frac{\pi}{4})}{\frac{1}{2}\gamma a^2} = 2 - \frac{\pi}{2}$$

$$\theta = \operatorname{atan}\left(2 - \frac{\pi}{2}\right) \cong 23^\circ$$ (6)

Approach using direct integration, that yields identical results, but requires abilities to set up integrals and complete integration

Problem 2

Hydrostatic pressure: $p = \rho g h = \gamma(a - a\sin\theta) = \gamma a(1-\sin\theta)$

Normal vector of quarter-circle surface: $\mathbf{n} = (n_x, n_y) = (\cos\theta, -\sin\theta)$

(a) Horizontal force on quarter-circle surface:

$$F_x = \int_A p n_x \mathrm{d}A = \int_0^{\pi/2} \gamma a^2 (1-\sin\theta)\cos\theta\,\mathrm{d}\theta = \gamma a^2 \int_0^{\pi/2} (\cos\theta - \sin\theta\cos\theta)\mathrm{d}\theta = \gamma a^2 \left[\sin\theta - \frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\gamma a^2$$

Moment of F_x about A (negative sign means clockwise direction)

$$M_x = -\int_A y p n_x \mathrm{d}A = -\int_0^{\pi/2} \gamma a^3 (1-\sin\theta)\cos\theta\sin\theta\,\mathrm{d}\theta = -\gamma a^3 \int_0^{\pi/2} (\cos\theta\sin\theta - \cos\theta\sin^2\theta)\mathrm{d}\theta$$

$$= -\gamma a^3 \left[\frac{1}{2}\sin^2\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = -\frac{1}{6}\gamma a^3$$

Line of action of F_x is along $y = -M_x / F_x = \frac{1}{6}\gamma a^3 / \frac{1}{2}\gamma a^2 = \frac{1}{3}a$

(b) Vertical force on quarter-circle surface:

$$F_y = \int_A p n_y \mathrm{d}A = -\int_0^{\pi/2} \gamma a^2 (1-\sin\theta)\sin\theta\,\mathrm{d}\theta = -\gamma a^2 \int_0^{\pi/2} (\sin\theta - \frac{1-\cos 2\theta}{2})\mathrm{d}\theta$$

$$= -\gamma a^2 \left[-\cos\theta - \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = -\gamma a^2 \left(1 - \frac{\pi}{4}\right)$$

Moment of F_y about A

$$M_y = -\int_A x p n_y \mathrm{d}A = -\int_0^{\pi/2} \gamma a^3 (1-\cos\theta)(1-\sin\theta)\sin\theta\,\mathrm{d}\theta$$

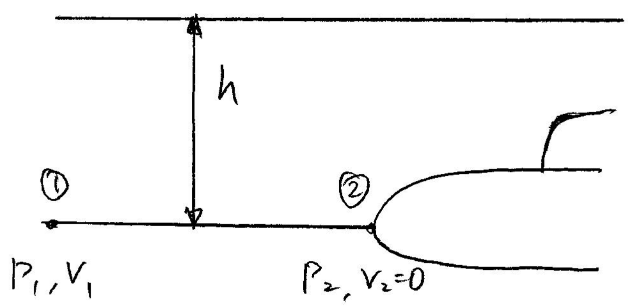
$$= -\gamma a^3 \int_0^{\pi/2} (\sin\theta - \sin^2\theta - \sin\theta\cos\theta + \sin^2\theta\cos\theta)\mathrm{d}\theta$$

$$= -\gamma a^3 \left[-\cos\theta - \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta - \frac{1}{2}\sin^2\theta + \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = -\gamma a^3 (\frac{5}{6} - \frac{\pi}{4})$$

Line of action of F_y is along $x = -M_y / F_y = \gamma a^3 (\frac{5}{6} - \frac{\pi}{4}) / \gamma a^2 \left(1 - \frac{\pi}{4}\right) = \frac{\frac{5}{6} - \frac{\pi}{4}}{1 - \frac{\pi}{4}} a = \frac{10 - 3\pi}{12 - 3\pi} a$

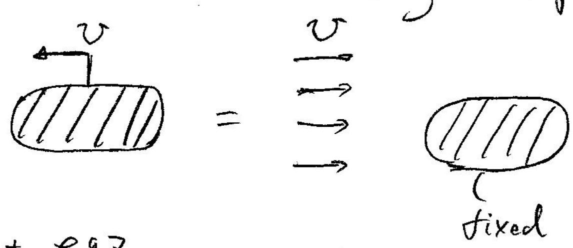
Problem 3

(a) Assume no waves.



Point ① far upstream of the submarine

Point ② at the stagnation point.



$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

$$V_2 = 0, \quad z_1 = z_2 \quad V_1 = v$$

stagnation pressure $\quad P_2 - P_1 = \underline{\underline{\frac{1}{2}\rho v^2}}$ $\quad$ 15

*Alternatively, it is also ok to give the total pressure at ②

$$P_2 = P_1 + \frac{1}{2}\rho v^2 = \rho g h + \frac{1}{2}\rho v^2$$

(b) $h = 10\,m$, $\quad v = 20$ knots $= 10.29\,m/s$

$\rho_{sw} = 1027\,kg/m^3$ $\quad$ ($\pm 3\,kg/m^3$ ok)

Stagnation pressure $\quad P_2 - P_1 = 54.4\,kPa$. $\quad$ 10

It is also ok to give the total pressure

$P_2 = 155.1\,kPa$