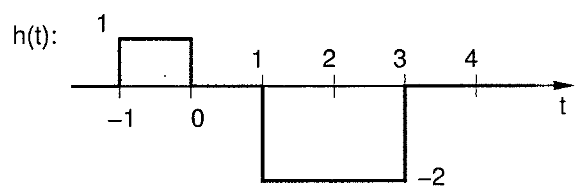
Key.

Problem 1 LTI Properties (22 pts)

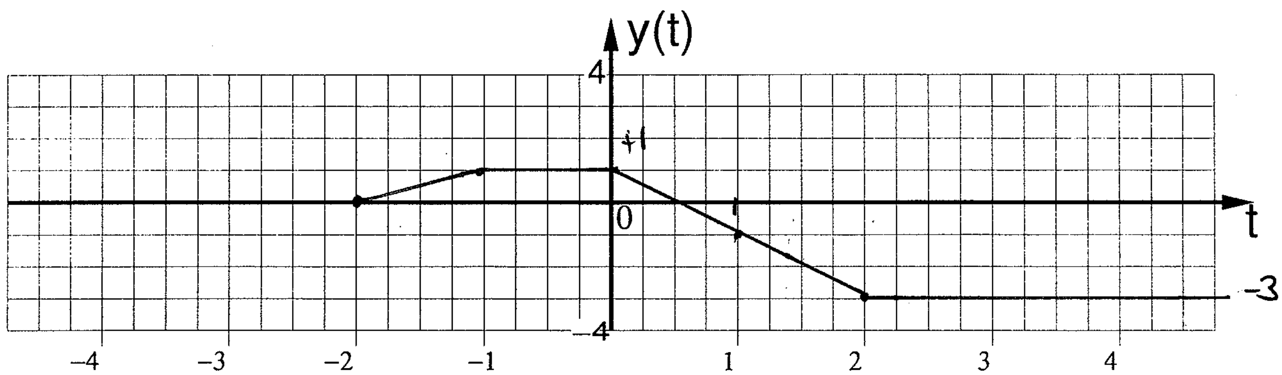
[16 pts] a. Classify the following systems, with input $x(t)$ and output $y(t)$. In each column, write "yes", "no", or "?" if the property is not decidable with the given information. (+1 for correct, 0 for blank, -0.5 for incorrect).

System	Causal	Linear	Time-invariant	BIBO
a. $y(t) = x(t) \cdot \Sigma_{n=-\infty}^{\infty} \delta(t-2n)$	yes	yes	no	no
b. $y(t) = x(t) * \Sigma_{n=0}^{\infty} \delta(t-2n)$	yes	yes	yes	no
c. $y(t) = x(t) - \frac{1}{2}\frac{dx(t+1)}{dt}$	no	yes	yes	no
d. $y(t) = \int_{-1}^{1} x(\tau)x(t-\tau)d\tau$	no	no	No	yes

[6 pts] e. An LTI system has impulse response $h(t)$ as shown below:



Given input $x(t) = u(t+1)$. Sketch the output $y(t)$ on the grid below, noting key times and amplitudes.



d. BIBO? integrate product $-1 < \tau < 1 \Rightarrow$ bounded.

Causal?

$y(t=1/2) = \int_{-1}^{1} x(\tau)\, x(1/2 - \tau)\, d\tau$, so $y(t=1/2)$ depends on $x(\tau)$ from $1/2$ to $1 \Rightarrow$ non causal.

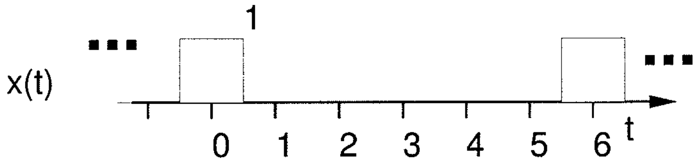
Time Invariant?

Consider $x(t) = \Pi(t)$ vs $x(t) = \Pi(t-3)$. not time invariant.

Key.

Problem 2 Fourier Series (25 pts)

You are given a periodic function $x(t)$ as shown, where the shape is a rectangular pulse of height 1 and width 1, centered at $t = 0$:



Note that $x(t)$ can be represented by a Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t},$$

where $a_k = \frac{\sin k\pi/6}{k\pi}$.

[1 pts] a. What is the fundamental frequency $\omega_o = \underline{\frac{2\pi}{T_0} = \frac{2\pi}{6}} = \pi/3$

[2 pts] b. What is the total time average power in $x(t)$? $\underline{1/6}$

$$\text{power} = \frac{1}{T}\int_T x^2(t)\,dt = \frac{1}{6}\int_{-1/2}^{1/2} 1^2 \cdot dt = 1/6$$

[5 pts] c. What is the percentage of the total power in $x(t)$ which is not at DC or the fundamental frequency?

percent = $\left(\frac{\frac{1}{6} - |a_0|^2 - 2\cdot|a_1|^2}{1/6}\right)100\%$

power at DC ($k=0$) $= a_0^2 = \frac{1}{36}$

power at fundamental ($k=1$) $= |a_1|^2 + |a_{-1}|^2 = 2\cdot\left(\frac{1}{2\pi}\right)^2 = \frac{1}{2\pi^2}$

total power $= \sum_{k=-\infty}^{\infty} |a_k|^2$

$a_0 = \frac{1}{6}$

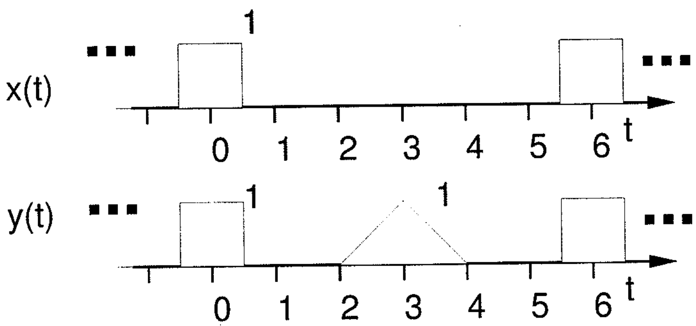
$a_1 = a_{-1} = \frac{\sin \pi/6}{\pi} = \frac{1}{2\pi}$

$$= \frac{\frac{1}{6} - \frac{1}{36} - \frac{1}{2\pi^2}}{1/6} = \left(1 - \frac{1}{6} - \frac{3}{\pi^2}\right)\cdot 100\%$$

(with calculator $\approx 53\%$).

Key.

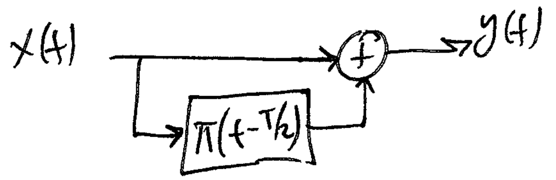
Problem 2, continued.
Given a new signal $y(t)$ as shown:



Periodic function $y(t)$ can be represented by a Fourier Series:

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_o t} \quad , \quad \omega_0 = \frac{2\pi}{6} = \pi/3 = \frac{2\pi}{T}$$

[12 pts] d. Find $b_k =$ ____________________



$$H(jk\omega_0) = 1 + \frac{2\sin\frac{k\omega_0}{2}}{k\omega_0} e^{-jk\omega_0 T/2}$$

$$= 1 + \frac{2\sin k\pi/6}{k\pi/3} e^{-j\pi k}$$

recall

$$e^{jk\omega_0 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(jk\omega_0)e^{jk\omega_0 t}$$

$$b_k = \frac{\sin k\pi/6}{k\pi}\left(1 + \frac{2\sin\frac{k\pi}{6}}{k\pi/3}(-1)^k\right).$$

thus $b_k = a_k H(jk\omega_0)$.

$$\Pi(t) \xrightarrow{\mathcal{F}} \frac{2\sin \omega/2}{\omega}$$

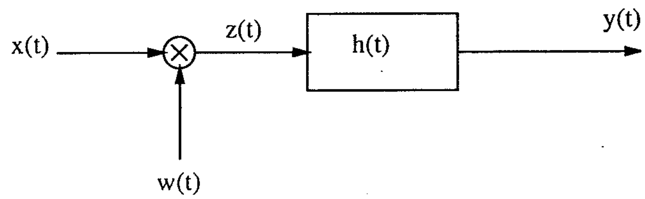
$$\Pi(t - T/2) \longrightarrow \frac{2\sin \omega/2}{\omega} \cdot e^{-j\omega T/2}$$

[5 pts] e. If $y(t) = x(t) * h(t)$, find $h(t) =$: $\delta(t) + \Pi(t - T/2)$

$$y(t) = x(t) + \Pi(t - T/2) * x(t)$$

Key.

Problem 3. Fourier Transform (26 pts)
For each part below, consider the following system:

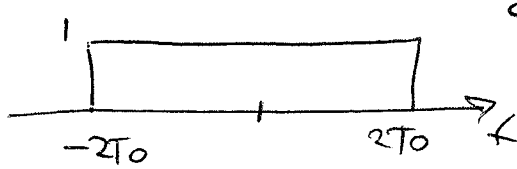


Where $x(t) = \cos(400\pi t) + \Pi(\frac{t}{4T_o})$, $\quad w(t) = \frac{1}{2T_o}\Pi(\frac{t}{2T_o})$, $\quad h(t) = \Sigma_{n=-\infty}^{\infty}\delta(t - \frac{n}{50})$ with $T_o = 1/100$ sec.

(Recall that $\Pi(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$.)

On the next page, sketch $Re\{X(j\omega)\}, Re\{Z(j\omega)\}, Re\{Y(j\omega)\}$ labelling height/area, center frequencies, and key zero crossings for $-500\pi \le \omega \le 500\pi$:

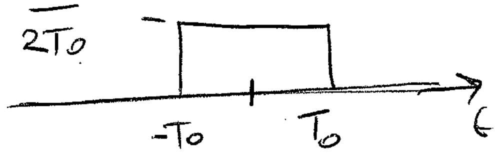
$\Pi(\frac{t}{4T_o}) =$



$\xRightarrow{\mathcal{F}} \dfrac{2\sin 2T_o\omega}{\omega}$

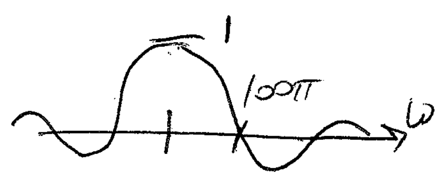
$= \dfrac{2\sin \omega/50}{\omega}$

$\Pi(\frac{t}{2T_o})$



$\xRightarrow{\mathcal{F}} \dfrac{1}{2T_o} \cdot \dfrac{2\sin T_o\omega}{\omega}$

$= \dfrac{100\sin \omega/100}{\omega}$



$z(t) = x(t)\, w(t)$

$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$

$= \frac{1}{2\pi}\left[\pi\delta(\omega - 400\pi) + \pi\delta(\omega + 400\pi)\right]$

$\quad * \; 100\,\dfrac{\sin(\omega/100)}{\omega}$.

$y(t) = z(t) * h(t)$.

$Y(j\omega) = Z(j\omega) \cdot H(j\omega)$

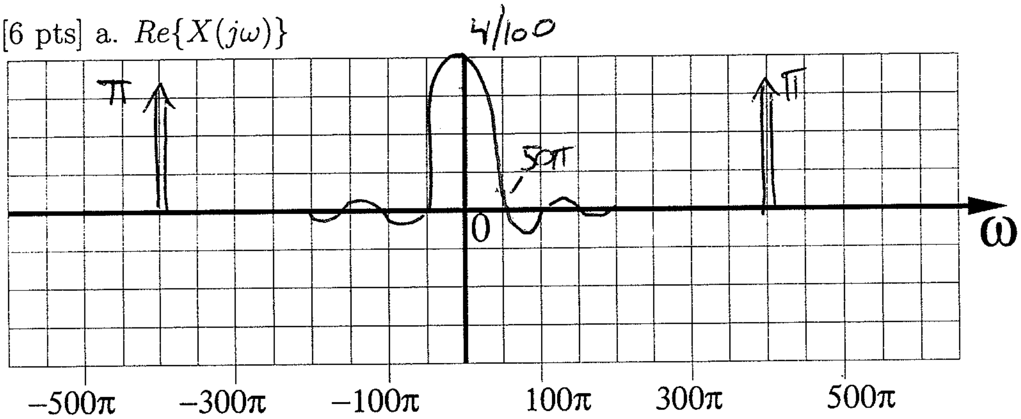
$H(j\omega) = \dfrac{2\pi}{1/50} \sum \delta(\omega - k\frac{2\pi}{1/50})$

$= 100\pi \sum \delta(\omega - 100k\pi)$.

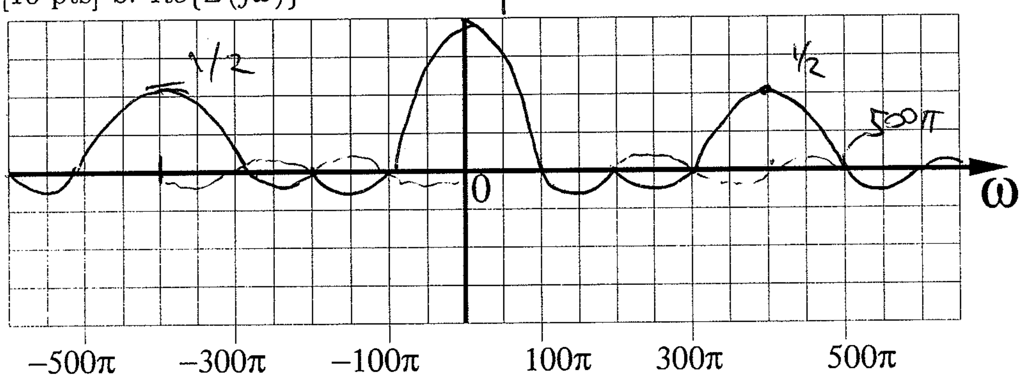
Key.

Problem 3, continued.

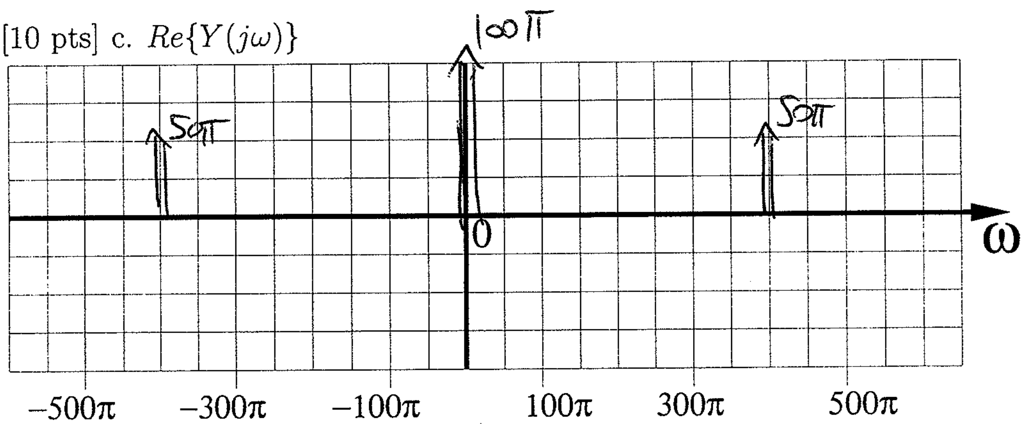
[6 pts] a. $Re\{X(j\omega)\}$



[10 pts] b. $Re\{Z(j\omega)\}$



[10 pts] c. $Re\{Y(j\omega)\}$



Key.

Problem 4. DTFT (27 points)

A causal LTI system with input $x[n]$ and output $y[n]$ is described by the transfer function:

$$H(e^{j\omega}) = \frac{j\sin\omega}{\cos\omega} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

[5 pts] a. Find the difference equation relating $y[n]$ and $x[n]$, corresponding to $H(e^{j\omega})$:

$y[n] =$ $x[n] - x[n-2] - y[n-2]$.

$$j X(e^{j\omega}) \sin\omega = Y(e^{j\omega}) \cos\omega$$

$$j X(e^{j\omega}) \frac{e^{j\omega} - e^{-j\omega}}{2j} = Y(e^{j\omega}) \frac{(e^{j\omega} + e^{-j\omega})}{2}$$

$x[n+1] - x[n-1] = y[n+1] + y[n-1]$

$y[n]$

LTI $\Rightarrow$ init cond's $= 0$, causal $h[n] = 0$ for $n < 0$

[7 pts] b. Find the impulse response $h[n]$, that is, the time response of the system to input $x[n] = \delta[n]$.

$$h[n] = \begin{cases} 0 & n \text{ odd} \\ 1 & n = 0 \\ 2(-1)^{n/2} & n \text{ even.} \end{cases}$$

n	$x[n]$	$x[n-2]$	$y[n]$	$y[n-2]$
0	1	0	1	0
1	0	0	0	0
2	0	1	-2	1
3	0	0	0	0
4	0	0	2	-2
5	0	0	0	0
6	0	0	-2	2

⋮

[10 pts] c. If $x[n] = 2\cos(\frac{\pi n}{3})$ find $y[n]$. $y[n] =$ $-2\sqrt{3}\sin \pi n/3$

$x[n] = e^{j\pi n/3} + e^{-j\pi n/3}$ (eigen functions).

$y[n] = H(e^{j\pi/3}) e^{j\pi n/3} + H(e^{-j\pi/3}) e^{-j\pi n/3}$

$H(e^{j\pi/3}) = \frac{j\sin\pi/3}{\cos\pi/3} = \frac{j\sqrt{3}/2}{1/2} = j\sqrt{3}$

$H(e^{-j\pi/3}) = \frac{j\sin(-\pi/3)}{\cos\pi/3} = -j\sqrt{3}$

$y[n] = j\sqrt{3}\left(e^{j\pi n/3} - e^{-j\pi n/3}\right)$

$= -2\sqrt{3}\sin \pi n/3$

Key

Problem 4, continued.

[5 pts] d. Let $z[n] = \cos[\frac{\pi n}{4}]\cos[\frac{\pi n}{2}]$. Find the DTFT of $z[n]$.

$Z(e^{j\omega}) =$ ______________________

$$z[n] = \frac{1}{4}\left(e^{j\pi n/4} + e^{-j\pi n/4}\right)\left(e^{j\pi n/2} + e^{-j\pi n/2}\right)$$

$$= \frac{1}{4}\left[e^{j\pi n\, 3/4} + e^{-j\pi n\, 3/4} + e^{j\pi n/4} + e^{-j\pi n/4}\right]$$

$$Z(e^{j\omega}) = \frac{1}{4}\cdot 2\pi\left(\delta(\omega - 3/4\,\pi) + \delta(\omega + \tfrac{3}{4}\pi) + \delta(\omega - \pi/4) + \delta(\omega + \pi/4)\right).$$

for $0 \le \omega < 2\pi$

ω periodic with period 2π.