Question 1. (16 points) Choose the correct answers, worth 2 points each. No justification necessary. Incorrect answers carry a 2-point penalty, so random choices are not helpful. You may leave any question blank for 0 points.

	F	For two $n \times n$ matrices A, B, $\mathrm{Det}(A - B) = \mathrm{Det}(A) - \mathrm{Det}(B)$.
	F	Two similar matrices have exactly the same eigenvectors.
T		If the columns of a real 2×2 matrix $M \neq I_2$ are orthonormal, then M is the matrix of a rotation or of a reflection in $\mathbf{R}^2$.
T		If $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
T		The columns of an $n \times n$ matrix S span $\mathbf{R}^n$ if and only if $\mathrm{Det}(S) \neq 0$.
T		The transpose of an orthogonal matrix is also an orthogonal matrix.
T		The matrix representing the reflection about a line through $\mathbf{0}$ in $\mathbf{R}^2$ is always diagonalizable over the reals.
	F	The left nullspace of the square matrix A is non-zero if and only if 0 is not an eigenvalue of A.
T		If A and B are orthogonal matrices of the same size, then AB^{-1} is an orthogonal matrix.
	F	If the columns of the matrix A are non-zero and pairwise orthogonal, then the system $A\mathbf{x} = \mathbf{b}$ must be consistent.
	F	The determinant of a real, square matrix is equal to the product of its real eigenvalues, included with their multiplicities.
T		Similar matrices have the same characteristic polynomial.

Question 2. (18 points) Select the correct answer for each question, worth 3 points each. There is no penalty for incorrect answers (but, of course, you forfeit the points).
Answer key: 1c, 2c, 3f, 4a, 5c, 6e, 7c, 8c, 9f

1. Under which circumstances is the square matrix A guaranteed to have non-zero determinant?

   (a) A has no zeroes on the diagonal
   (b) A is the coefficient matrix of a consistent linear system
   (c) The linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution
   (d) A has only positive entries
   (e) A has orthogonal columns
   (f) None of the above

2. We are told that the 4×4 real matrix A has exactly two distinct real eigenvalues, with multiplicity one each. We can safely conclude that

   (a) A is invertible
   (b) A is diagonalizable over $\mathbf{R}$
   (c) A is diagonalizable over $\mathbf{C}$
   (d) A is symmetric
   (e) A is orthogonal
   (f) None of the above

3. If $\frac{1}{2}\left(\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2\right) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ for two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbf{R}^n$, we can conclude

   (a) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
   (b) $\mathbf{u} = \mathbf{v} = \mathbf{0}$
   (c) $\|\mathbf{u}\| = \|\mathbf{v}\|$
   (d) $\mathbf{u}, \mathbf{v}$ are linearly dependent
   (e) $\mathbf{u}, \mathbf{v}$ must be vectors in $\mathbf{R}^2$
   (f) Nothing: that is always true

4. In a new basis of $\mathbf{R}^2$, the coordinate vector of the vector $\begin{bmatrix}1\\2\end{bmatrix}$ is $\begin{bmatrix}2\\3\end{bmatrix}$ and that of the vector $\begin{bmatrix}3\\4\end{bmatrix}$ is $\begin{bmatrix}4\\5\end{bmatrix}$. Then, the coordinate vector of $\begin{bmatrix}5\\6\end{bmatrix}$ is

(a) $\begin{bmatrix}6\\7\end{bmatrix}$ (b) $\begin{bmatrix}6\\8\end{bmatrix}$ (c) $\begin{bmatrix}5\\6\end{bmatrix}$

(d) $\begin{bmatrix}5\\7\end{bmatrix}$ (e) $\begin{bmatrix}7\\6\end{bmatrix}$ (f) Cannot be determined from the data

5. A least-squares solution $\hat{\mathbf{x}}$ of a (possibly inconsistent) $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ is characterized by

(a) $\hat{\mathbf{x}}$ is the shortest vector in $\mathrm{Col}(A)$

(b) $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$

(c) $\hat{\mathbf{x}}$ is in $\mathbf{R}^n$ and $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ is as short as possible

(d) $\hat{\mathbf{x}}$ is in $\mathrm{Col}(A)$ and $\|\hat{\mathbf{x}} - \mathbf{b}\|$ is as short as possible

(e) $A\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Row}(A)$

(f) A least squares solution may not exist so we cannot safely characterize it

6. Under which of the following assumptions is the real $n \times n$ matrix A guaranteed to be diagonalizable over the reals?

(a) Row-reduction of A finds n distinct pivots

(b) A is an orthogonal matrix

(c) A has n distinct eigenvalues

(d) A has n real eigenvalues, with multiplicities counted

(e) There are n linearly independent real eigenvectors for A

(f) All eigenvalues of A are real and there is at least one eigenvector for each

7. Pick the matrix on the list which is NOT diagonalizable (not even over $\mathbf{C}$), if any; else, pick option (f).

(a) $\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}$ (b) $\begin{bmatrix}2 & 1\\-1 & 1\end{bmatrix}$ (c) $\begin{bmatrix}2 & 1\\-1 & 0\end{bmatrix}$

(d) $\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ (e) $\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}$ (f) All of them are diagonalizable over $\mathbf{R}$ or $\mathbf{C}$

8. If A is an $m \times n$ matrix and B an $n \times m$ matrix, then $\mathrm{Det}(AB) = \mathrm{Det}(BA)$, provided that the following additional condition holds: (select the *least restrictive* correct answer)

(a) Always, no condition needed

(b) $m = n$ and both A and B are invertible

(c) $m = n$

(d) $m = n$ and one of A, B is invertible

(e) $AB = BA$

(f) A and B are in echelon form

9. We are told that the real, square matrix A is diagonalizable over $\mathbf{R}$. Pick the matrix from the list which is NOT guaranteed to be diagonalizable over $\mathbf{R}$, if any; or else, pick option (f).

(a) A^2 (b) A^TA (c) A^{-1}, assuming A invertible

(d) A^T (e) $A + I$ (f) All of them are diagonalizable

Question 3. (17 points, 6+5+6)
For the matrix A below,

1. Find its characteristic polynomial
2. Find its eigenvalues. (*Note:* there will be complex eigenvalues.)
3. Find a (non-zero) eigenvector for each eigenvalue.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 5 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

1.
$$\chi_A(t) = \det \begin{bmatrix} 3-t & -1 & 1 \\ 5 & -1-t & 3 \\ 0 & 0 & 2-t \end{bmatrix} = (2-t) \cdot \det \begin{bmatrix} 3-t & -1 \\ 5 & -1-t \end{bmatrix} = (2-t)(t^2 - 2t + 2)$$

2. The roots of $\chi_A(t)$ are $\lambda = 2$ and $\mu_\pm = 1 \pm i$. So A has one real and a pair of complex-conjugate, non-real eigenvalues.

3. For $\lambda = 2$, $\mathrm{Nul}(A - 2I_3)$ is spanned by $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. For μ_+, we seek a non-zero complex vector $\mathbf{w}_+$ in

$$\mathrm{Nul} \begin{bmatrix} 2-i & -1 & 1 \\ 5 & -2-i & 3 \\ 0 & 0 & 1-i \end{bmatrix};$$

clearly we need the third entry of $\mathbf{w}_+$ to vanish, and the top two entries must verify

$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{0}$$

so $w_1 = 1, w_2 = 2 - i$ will work and we can take $\mathbf{w}_+ = \begin{bmatrix} 1 \\ 2-i \\ 0 \end{bmatrix}$. We can get a $\mathbf{w}_-$ by complex conjugation, $\mathbf{w}_- = \begin{bmatrix} 1 \\ 2+i \\ 0 \end{bmatrix}$.

Question 4. (14 points, 8+6)

Let $V \subset \mathbf{R}^4$ be the span of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

(a) Use the Gram-Schmidt process on these vectors to produce an orthogonal basis of V.
(b) Find the orthogonal projection of the vector $[1, 0, 0, 1]^T$ onto V. Explain your method.
(a) We take $\mathbf{u}_1 = [1, -1, 0, 0]^T$. Then,

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{[1, 0, -1, 0] \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{[1, 0, 0, -1] \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{[1, 0, 0, -1] \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}.$$

(b) It is easier to project onto the orthogonal complement of V, which is the line spanned by $\mathbf{a} = [1, 1, 1, 1]^T$. The projection of $[1, 0, 0, 1]^T$ onto that line is $\frac{[1,0,0,1]\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{1}{2}\mathbf{a}$, so the orthogonal projection of $[1, 0, 0, 1]^T$ onto V is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\mathbf{a} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Question 5. (10 points)

Set up a consistent system of linear equations for the vector $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$, whose solution gives the best fit of the data points

t	-2	-1	0	1	2
y	1	0	1	2	2

to the relation $y = s_1 t + s_2 t^2 + s_3 t^3$, in the sense of least squares. Explain your procedure, including what your solution is attempting to achieve. (You need not solve the system explicitly)

We are trying to solve the (inconsistent) system

$$\begin{cases} -2s_1 + 4s_2 - 8s_3 = & 1 \\ -s_1 + s_2 - s_3 = & 0 \\ 0 + 0 + 0 = & 1 \\ s_1 + s_2 + s_3 = & 2 \\ 2s_1 + 4s_2 + 8s_3 = & 2 \end{cases}, \qquad \text{or} \qquad \begin{bmatrix} -2 & 4 & -8 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix} \cdot \mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Calling A the 5×3 matrix and $\mathbf{b}$ the 5-component vector of values, we want to find the least squares solution of this system, the one which minimizes the error $\sum_{t=-2}^{2} |y(t) - s_1 t - s_2 t^2 - s_3 t^3 = \|A\mathbf{s} - \mathbf{b}\|^2$.
For this, $A\mathbf{s} - \mathbf{b}$ must be perpendicular to Col(A) and this condition is encoded by the normal equations

$$A^T A\mathbf{s} = A^T\mathbf{b}$$

which in this case give

$$\begin{bmatrix} 10 & 0 & 34 \\ 0 & 34 & 0 \\ 34 & 0 & 130 \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 10 \end{bmatrix}$$

(The solution is $s_1 = 1.25, s_2 = 7/17, s_3 = -.25$)