Solutions for Midterm 2

Question 1 [Typos corrected as announced in class]

T		Eigenvectors for distinct eigenvalues of a real symmetric matrix are orthogonal
T		The determinant of any lower-triangular square matrix is the product of the diagonal entries
	F	The eigenvalues of a non-singular matrix are all real (correct: non-zero)
	F	If A and B are square matrices and $\det A = 1, \det B = 2$, then $\det(A+B) = 3$
T		A change-of-coordinates matrix is always invertible
	F	Any orthogonal matrix is diagonalizable over $\mathbf{R}$ (only reflections are)

T		The determinant of an orthogonal matrix is always ± 1
T		For an $m \times n$ matrix A, vectors in the column space of A are orthogonal to vectors in the left nullspace
T		If a real symmetric 4×4 matrix has exactly two eigenvalues, then one of the eigenspaces has dimension 2 or more
T		If A is an orthogonal matrix, then the linear map $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto.
	F	If every row of the square matrix A is a linear combination of the rows of the square matrix B, then $\det A = \det B$ (false, e.g. take $A = 0$, $B = I_n$)
T		If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals that from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u} \perp \mathbf{v}$

	F	If a 2×2 matrix is diagonalizable, then it has distinct eigenvalues (counterexample, I_2)
T		If $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
T		If $A = SBS^{-1}$ then A and B have the same characteristic polynomial.
T		If the columns of the 5×3 matrix A are orthonormal, then the linear transformation $\mathbf{x} \mapsto A \cdot \mathbf{x}$ preserves lengths and angles
T		The determinant of a square matrix is $\pm$ the product of the pivots
	F	If A is a 4×3 matrix with orthonormal columns, then $A^T A$ is the orthogonal projection matrix on Col(A) (that would be AA^T)

Comments:

- There seems to have been some confusion about the definition of pivots across sections; the question about determinant and pivots will therefore be discarded from the grading.

Question 2
See the proofs under space Resources

Question 3
The easiest way is to construct the projection $P^\perp$ onto the normal line, spanned by the vector $[1,-1,1]^T$:

$$P^\perp = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Then,

$$P = I_3 - P^\perp = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Alternatively, we find a basis of the subspace – the usual one would be $[1,1,0]^T, [-1,0,1]^T$ and make it orthogonal using Gram-Schmid. The second vector becomes

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{[1,1,0]\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

The second vector has square-length 2/3. From here the formula for P is

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -1/2 & 1/2 & 1 \end{bmatrix}\cdot\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

which works out to the same answer.

For the second part, computing the characteristic polynomial is possible but painful. But, geometrically, P leaves unchanged the vectors in the plane $x - y + z = 0$: so all those vectors are 1-eigenvectors. Also, P sends all vectors in the normal line to 0, so that line is in the nullspace. We are done, because we spotted three independent eigenvectors:

P has two eigenvalues, 0 and 1, with eigenspace dimensions 1 and 2.

Question 4 The characteristic polynomial of A is $\lambda^2 - 2\lambda + 1$, with the double root $\lambda = 1$. Now,

$$A - I_2 = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$$

which has a 1-dimensional nullspace spanned by the vector $[2,-3]^T$. So, no, A is not diagonalizable because there is no second independent eigenvector. If we take $\mathbf{v} = [2,-3]^T$, we see that $A\mathbf{v} = \mathbf{v}$, $A^2\mathbf{v} = \mathbf{v}, \ldots A^n\mathbf{v} = \mathbf{v}$ for all n. So that choice works. (It can be shown that only multiples of this $\mathbf{v}$ work, but you were not asked to check that.)

Question 5

The system of equations in the unknown coefficients a, b that we must solve by least squares is

$$\begin{cases} -a+b &= 1 \\ 0\cdot a+b &= 1 \\ a+b &= 0 \\ 2a+b &= 2 \end{cases}$$

The relevant coefficient matrix is

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \text{so} \quad A^T = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}.$$

The normal equations are

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

with the solution $a = 0.2, b = 0.9$; so $\ell(x) = 0.2x + 0.9$.