Mathematics 1A, Fall Semester 2003
Instructor: Garth Dales
December 16, 2003

Final Examination

Your Name: ______________________________

Your SID: ______________________________

Your GSI's Name and Section Time: ______________________________

Directions:

- Do not open your exam until your are instructed to do so.
- You may not use any external aids during the exam: NO books, NO lecture notes, NO formula sheets, NO cell phones, NO calculators.
- Answers without explanation will not receive credit. *You must show and justify your work.* If necessary, use the backs of the pages or the extra pages attached to your exam, and indicate you have done so.
- When time is called, you must stop working and close your exam.
- All questions are worth 20 points each (for a total of 200 points).

Good Luck!

Score			
1.(a)	(b)	(c)	
2.(a)	(b)		
3.(a)	(b)	(c)	
4.			
5.(a)	(b)	(c)	(d)
6.(a)	(b)	(c)	(d)
7.(a)	(b)		
8.(a)	(b)		
9.			
10.(a)	(b)	(c)	
Total	/200		

1. (a) *(6 points)* State (without proof) the *Squeeze Rule* that gives a condition for $\lim_{x \to a} g(x) = \ell$ for a certain function g defined near a.

(b) *(6 points)* Evaluate the following limit, if it exists. Be sure to justify your answer.

$$\lim_{t \to 0} t \sin(1/t)$$

(c) *(8 points)* Use l'Hôpital's rule to evaluate the following limit:

$$\lim_{x \to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3} .$$

2. (a) For each of the following functions f, calculate its derivative f'.

i. *(4 points)*

$$f(x) = \int_3^x \sin(t^3 + t)\,dt$$

ii. *(4 points)*

$$f(x) = \tan^{-1}(x^2 + 1)$$

iii. *(5 points)*

$$f(x) = (\cosh x)^{\log x}$$

(b) *(7 points)* Find the equation of the tangent line to the curve specified by the equation

$$3 - 2y^3 = 6x^2y$$

at the point (a, b). Find a point (a, b) on the curve at which the tangent is parallel to the line $x + y = 0$.

3. (a) *(4 points)* Write down the *Principle of Mathematical Induction*, which involves certain statements P_n for $n = 1, 2, 3, \ldots$.

(b) *(6 points)* Prove that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

for each $n \in \mathbb{N}$.

(c) *(10 points)* Use the definition of area (involving Riemann sums) to show that

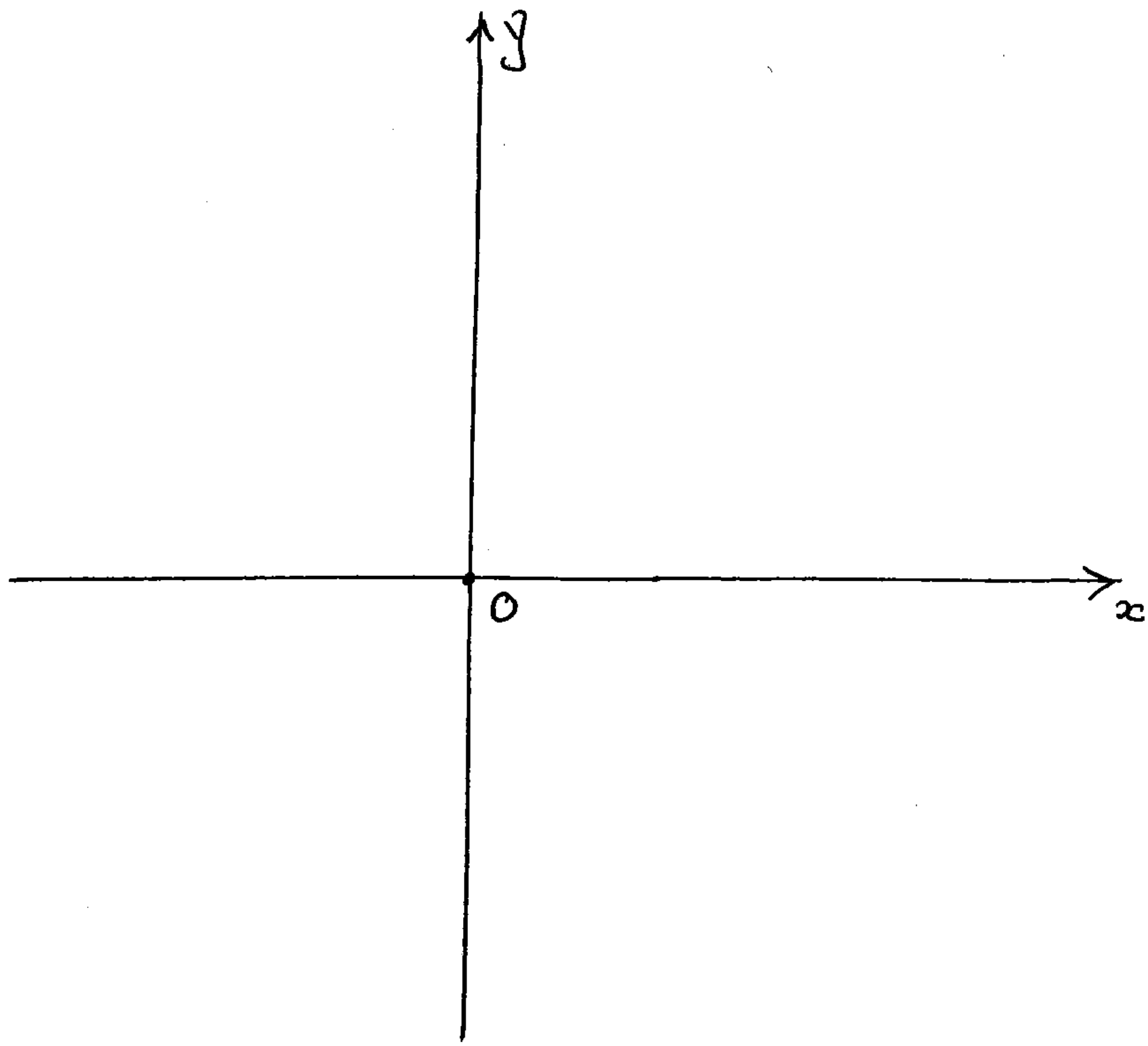
$$\int_0^1 x^2\,dx = \frac{1}{3}\,.$$

[Note: You will receive no credit for applying the Fundamental Theorem of Calculus.]

4. *(20 points)* Sketch the graph of the function

$$y = \frac{e^x}{x}$$

by first determining the (maximal possible) domain, any intercepts with the axes, asymptotes, any local maxima and minima, intervals of increase/decrease, concavity, and inflection points. Coordinate axes are provided on the next sheet.


y
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x

5. *(20 points)* Determine whether each of the following statements is true or false. If true, explain why. If false, give a counter-example, and explain why this counter-example contradicts the statement.

   A wrong answer will receive zero points. A blank answer will receive 1 point. A correct answer (either "true" or "false") with no explanation will receive 2 points. A correct answer with a full explanation will receive 5 points.

   (a)

$$\int_{-2}^{1} \frac{1}{x^4}\, dx = -\frac{3}{8}$$

   (b) All continuous functions have antiderivatives.

(c) If f and g are continuous functions on $[a,b]$, then

$$\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

(d) If f and g are differentiable and $f(x) \leq g(x)$ on $[a,b]$, then $f'(x) \leq g'(x)$ on $[a,b]$.

6. Evaluate the following (definite or indefinite) integrals.

(a) *(5 points)*

$$\int \frac{dx}{9x^2+1}$$

(b) *(5 points)*

$$\int_0^3 (x^3 - 6x)\,dx$$

(c) *(5 points)*

$$\int_0^a x\sqrt{x^2+a^2}\,dx, \quad \text{where } a>0$$

(d) *(5 points)*

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

7. (a) *(12 points)* Let f be a continuous function on the closed interval $[a, b]$. For each x with $a \leq x \leq b$, define

$$F(x) = \int_a^x f(t)\,dt\ .$$

Prove that F is differentiable at x for $a < x < b$, and that $F'(x) = f(x)$.

(b) *(8 points)* Define

$$\log x = \int_1^x \frac{dt}{t} \quad \text{for } x > 0.$$

Establish the following properties of $\log x$ *just by using this definition:*

i) $\log x$ is increasing and concave down on $(0, \infty)$.

ii) $\log(xy) = \log x + \log y \quad$ for each $x, y > 0$.

8. (a) *(8 points)* Find the area of the region bounded by the parabolas

$$y = x^2 \quad \text{and} \quad y = 2x - x^2.$$

(b) *(12 points)* A curve in the plane is defined by the equation

$$y^2 = x^2(x+3).$$

Give a rough sketch of the graph of this curve, showing intersections with the axes. (Do not compute derivatives or consider concavity.) Determine the area enclosed by the loop of the curve.

9. *(20 points)* Compute the volume of the solid obtained by rotating the region enclosed by the curves

$$y = x, \quad x = y^3$$

about the line $x = 1$.

10. (a) *(6 points)* Let f be a function that is differentiable at $x = a$. Write down the definition of $f'(a)$ in terms of a limit, and write down the formula for the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

(b) *(6 points)* Give a formula for a function f with domain $\mathbb{R}$ such that f is continuous at every point of $\mathbb{R}$ and such that f is differentiable at all points of $\mathbb{R}$ except for $x = -1$, $x = 0$, and $x = 1$, where it is not differentiable.

(c) *(8 points)* Prove by contradiction that there is no rational number x with $x^2 = 2$. (Recall that a rational number has the form p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.)