First midterm of Math 54, Fall 2012

Sep 13th 8:00-9:30am

Name KEY

SID 1-3373

Section Jason; 102 & 107

Note:

(1) *This is a closed book exam. No notes either. No calculators are allowed. All cellphones and other electronic devices should be off.*

(2) *Answers without explanation will not be accepted. Make sure to include your reasoning to get full credit. Write neatly and if you need extra space use the back of the pages containing the problems. If you do this make sure that the continuation is marked clearly.*

(3) *This exam contains 4 problems, each problem is worth the number of points indicated at the beginning of the problem.*

(4) *Read all questions carefully before you start trying to answer them.*

1	
2	
3	
4	
Total	

1. (20 points) The following statements are equivalent, except for one that is not. Identify it. Give an example showing that your choice is different from the other three.

   Let $\mathbf{A}$ be an $n \times n$ matrix.

   (a) The columns of $\mathbf{A}$ do not span $\mathbf{R^n}$.

   (b) The equation $\mathbf{Ax} = \mathbf{b}$ cannot be solved for all $\mathbf{b}$ in $\mathbf{R^n}$.

   $\boxed{\text{(c)}}$ The linear transformation that sends an arbitrary $\mathbf{x}$ into $\mathbf{Ax}$ is one-to-one.

   (d) The reduced row-echelon form of $\mathbf{A}$ has one row consisting of all zeros.

Solution 1: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. – Then $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, so that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in the span of the columns of A, so $\boxed{\text{(a) is true for } A}$.

– Similarly, $A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ can't be solved, so $\boxed{\text{(b) is true for } A}$.

– $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\vec{x} \mapsto A\vec{x}$ is not one-to-one, so $\boxed{\text{(c) is false for } A}$.

– A is already in rref, and it has a row of zeroes, so $\boxed{\text{(d) is true for } A}$.

So $\boxed{\text{(c)}}$ is different.

Solution 2: (a), (b), and (d) are equivalent for any matrix A, even a non-square matrix, and they all mean that A does not have a pivot in every row.

For any matrix A, (c) means that A has a pivot in every column.

For square matrices, "pivot in every row" is the same as "pivot in every column." That means

(a), (b), the exact opposite of (c), and (d)

are all equivalent, so $\boxed{\text{(c)}}$ is different and $\boxed{\text{EVERY square matrix works as an example.}}$

Warning: The problem specifically asks for an example. To get full credit, you must give an example and explain why, for your example, (c) is different from (a), (b), and (d).

2. (30 points) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

find all solutions to **Ax**=**b** for **b**=[3, 9, 13].

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 4 & 5 & 6 & 7 & 9 \\ 6 & 7 & 8 & 9 & 13 \end{array}\right] \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & -3 & -6 & -9 & -3 \\ 0 & -5 & -10 & -15 & -5 \end{array}\right]$$

$$\xrightarrow[R_3 \to -\frac{1}{5}R_3]{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So: $\boxed{(x_1, x_2, x_3, x_4) = (1 + x_3 + 2x_4,\ 1 - 2x_3 - 3x_4,\ \text{free},\ \text{free})}$

In parametric vector form: $\boxed{\vec{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ for any numbers } x_3 \text{ and } x_4}$

3. (25 points)

Given the three vectors $\mathbf{v_1} = (1, -3, 2)$, $\mathbf{v_2} = (2, -6, h)$ and $\mathbf{v_3} = (5, -7, 0)$ in $\mathbf{R}^3$

a) For what values of h is $\mathbf{v_3}$ in the span of the other two vectors?

b) For what values of h is the set of three vectors a linearly independent set?

$$\begin{bmatrix} 1 & 2 & 5 \\ -3 & -6 & -7 \\ 2 & h & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 8 \\ 0 & h-4 & 10 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \underbrace{\begin{bmatrix} 1 & 2 & 5 \\ 0 & h-4 & 10 \\ 0 & 0 & 8 \end{bmatrix}}_{\text{echelon form}}.$$

a) Treating the previous matrix as a 3x3 augmented matrix, the matrix always has a pivot in the augmented column, so is always inconsistent, so $\boxed{\text{no } h}$ work

b). Treating the previous matrix as a 3x3 coefficient matrix, the matrix has a pivot in every column, i.e. has independent columns, only when $h-4 \neq 0$, i.e. $\boxed{\text{all } h \neq 4}$ work.

4. (25 points)

Label the following statements true or false. In each case either give an explanation or a counterexample. Read all questions carefully.

i. Subsets of linearly dependent sets of vectors in $\mathbf{R}^n$ are linearly dependent

$\boxed{\text{FALSE}}$ Counterexample: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is dependent. (it contains the zero vector), but $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ is a subset that is independent (it's a single nonzero vector).

ii. Subsets of linearly independent sets of vectors in $\mathbf{R}^n$ are linearly independent <u>Prove using contrapositive:</u>

$\boxed{\text{TRUE}}$ Suppose $\{\vec{v}_1, \ldots, \vec{v}_p\}$ has a subset $\{\vec{w}_1, \ldots, \vec{w}_q\}$ that is dependent. Then one of the vectors $\vec{w}_i$ is a linear combination of the other $\vec{w}$'s. But since all of the $\vec{w}$'s are all also $\vec{v}$'s, this means that one of the $\vec{v}$'s is a lin. comb. of some of the other $\vec{v}$'s. So $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is dependent.

— There are many, many, other correct explanations, but all should have the same idea.

iii. Any homogeneous system of linear equations has at least one solution

$\boxed{\text{TRUE}}$ A homogeneous system is one of the form $A\vec{x} = \vec{0}_m$ for some $m \times n$ matrix A, some $\vec{x}$ in $\mathbb{R}^n$, and $\vec{0}_m$ is the zero vector in $\mathbb{R}^m$. But if $\vec{x} = \vec{0}_n$, the zero vector in $\mathbb{R}^n$, then

$$A \cdot \vec{0}_n = \vec{0}_m,$$

So $A\vec{x} = \vec{0}_m$ always has at least one solution, namely $\vec{x} = \vec{0}_n$.