1 True/False. [24 pts]

Circle one of the provided answers please!
No negative points will be assigned for incorrect answers.

(a) TRUE or FALSE: Given independent events $A, B$ where $A$ and $B$ have nonzero probability, then $A \cap B$ is nonempty.

(b) TRUE or FALSE: If $A, B,$ and $C$ are mutually independent, then $Pr[A|B, C] = Pr[A].$

(c) TRUE or FALSE: If $Pr[A|B] = 2Pr[A]$, then $Pr[B] > Pr[A].$

(d) TRUE or FALSE: It is necessarily true that the variance of a random variable $X$ is $\leq (E(X))^2.$

(e) TRUE or FALSE: It is necessarily true that the variance of a random variable $X$ is $\leq E(X^2).$

(f) TRUE or FALSE: For disjoint events $A$ and $B$, the $Pr[A \cap B] = Pr[A] \times Pr[B].$

(g) TRUE or FALSE: For independent events, $Pr[A \cup B] = Pr[A] + Pr[B].$

(h) TRUE or FALSE: For a Poisson random variable $X$ with parameter $\lambda$, the $Pr[X = i + 1] \leq Pr[X = i]$ for all $i \geq \lambda.$
(i) **True or False:** For a Poisson random variable $X$ with parameter $\lambda = 1$, then Chebyshev’s inequality ensures that the $Pr[X \geq 11] \leq \frac{1}{100}$.

(j) **True or False:** For a binomially distributed variable $X$ with parameter $p = \frac{1}{2}$ and $n = 100$, Chebyshev’s inequality ensures that the $Pr[X \geq 75] \leq \frac{1}{10}$.

(k) **True or False:** Given two random variables, $X$ with Poisson distribution and $Y$ with a geometric distribution, both with mean $\mu$, we can conclude that $E[X + Y] > E[2X]$.

(l) **True or False:** The maximum variance binomial distribution with parameter $n$ has parameter $p = 1$.

(m) **True or False:** Given a random variable $S = X_1 + \ldots + X_n$ where the $X_i$’s are chosen independently from the same distribution, and any $\alpha$, $Pr[|S - E[S]| \geq \alpha]$ goes to 0 as $n$ goes to infinity.
2 Short answer. [43 pts]

For parts a to b, consider a deck with just the four aces (red: hearts, diamonds; black: spades, clubs). Melissa shuffles the deck and draws the top two cards.

(a) [4 pts] Given that Melissa has the ace of hearts, what is the probability that Melissa has both red cards?

(b) [4 pts] Given that Melissa has at least one red card, what is the probability that she has both red cards?

(c) [4 pts] Suppose that A and B are independent, C is disjoint from both A and B and \( P[A] = P[B] = P[C] = \frac{1}{4} \). Compute \( P[A \cup B \cup C] \).

For parts d to h, we consider two events A and B such that \( P(A) = 0.3 \) and \( P(B) = 0.4 \). Compute \( P(A|B) \) in each of the following cases:

(d) [3 pts] A and B are independent

(e) [3 pts] A and B are disjoint

(f) [3 pts] A \implies B

(g) \( P[A \cap B] = 0.1 \)

(h) [3 pts] \( P(A \cup B) = 0.5 \)
(i) [4 pts] The number of accidents (per month) at a certain factory has a Poisson distribution. If the probability that there is at least one accident is $1/2$, what is the probability that there are exactly two accidents?

(j) [4 pts] A pair of dice is rolled until either a 4 is rolled (the numbers on the two dice add up to 4) or a 7 is rolled. What is the expected number of rolls needed?

(k) [4 pts] There is a test to determine whether one has boneitis, but the test is not always accurate. For those who do have boneitis, the test has an $4/5$ chance of coming out positive. For those who don’t have boneitis, the test has a $1/9$ chance of coming out positive. Overall, about 10% of people have boneitis.

Suppose the test comes out positive for That Guy. What is the probability That Guy has boneitis?

(l) [4 pts] A hand of 13 cards are chosen (without replacement) at random from a standard deck of 52 poker cards. What is the expected number of four-of-a-kinds that we see from these 13 cards? (No need to evaluate the expression to get a number.)

(Four-of-a-kind is four cards of the same rank. For example, the hand “A♠A♥A♦A♣K♣K♥K♦2♠2♥2♣6♦” contains three four-of-a-kinds, namely the aces, the kings and the twos.)
The probability that all the cards in a certain rank are included in a hand is \( \frac{48}{9} \times \frac{52}{13} \).

By linearity of expectation we get the expected number of 4-of-a kinds is \( 13 \times \frac{48}{9} \times \frac{52}{13} \).

An alternate way to do this is to define a random variable for each of \( \binom{13}{4} \) subsets of 4 cards from a 13-card hand and compute the probability that those 4 cards have of a single rank \( \frac{13}{52} \).

This gives \( \binom{13}{4} \times \frac{13}{52} \).

I would hope they give the same number.
3  3-SAT. [15 pts]

A 3-conjuctive normal form (CNF) formula is a boolean formula consisting of the “and” of a sequence of clauses where each clause consists of the “or” of three literals. For example, \( \phi = (x_1 \lor x_2 \lor \overline{x_5}) \land (x_5 \lor x_2 \lor x_1) \). (No variable can appear twice in a single clause.)

One wishes to find an assignment to the variables to maximize the number of true clauses. The literals work in the natural manner: \( x_i = T \) if and only if \( x_i = F \). In the example above, the assignment, \( x_1 = T, x_2 = T \) and \( x_5 = F \) satisfies one clause in \( \phi \), where \( x_1 = T, x_2 = F, x_5 = F \) satisfies two clauses in \( \phi \).

(a) For a particular formula with \( n \) clauses, consider choosing a random assignment to the variables, i.e., \( x_i = T \) or \( x_i = F \) with equal probability. What is the expected number of satisfied clauses?

(b) Let \( U \) be a random variable corresponding to the number of unsatisfied clauses. What is \( E(U) \)?

(c) Upper bound the probability that \( U \) is larger than \( (1 + \varepsilon)E(U) \) for \( \varepsilon \geq 0 \) as a function of \( \varepsilon \). (You should give a nontrivial bound here.)

(d) Consider repeating this experiment until one finds an assignment that leaves at most \( (1 + \varepsilon)E(U) \) unsatisfied clauses. Give an upper bound on the expected number of repetitions.
4 The evolution of a social network. [18 pts]

(We give a simplified analysis of the connectivity of a social network.)

Say one person in a class of $n$ people knows a secret, perhaps where the midterm is. Occasionally a randomly chosen person $A$ who doesn’t know the secret calls a randomly chosen person $B$ ($B \neq A$) and learns the secret if $B$ knows it.

Let $X_2$ be a random variable that represents the number of calls (no two calls are simultaneous) until two people know the secret.

(a) What is the distribution of $X_2$?

(b) What is $E[X_2]$?

(c) Let $X_i$ be the number of calls needed to go from $i-1$ people knowing the secret to $i$ people. What is $E[X_i]$?

(d) What is the expected time for everyone to know the secret?
(e) Bound your expression to within a constant factor for large $n$. Your expression should not have a summation. (You may use $\Theta(\cdot)$ notation, recall that $2n^2 - 5n + 2 = \Theta(n^2)$.)