Math 54. **Second Midterm**

1. (12 points) A chemist solves a nonhomogeneous system of seven linear equations in ten unknowns and finds that four of the unknowns are free variables. Can the chemist be certain that, if the right-hand sides of the equations are changed, the new nonhomogeneous linear system will have a solution? Explain.

2. (21 points) The sets

$$\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

are bases of a vector subspace V of $\mathbb{R}^3$.

(a). Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

(b). Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

(c). If $T\colon V \to V$ is a linear transformation whose $\mathcal{B}$-matrix is $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then find $[T]_{\mathcal{C}}$.

3. (15 points) For each of the following matrices, either show that it can be diagonalized, or that it can't be diagonalized.

(a). $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

4. (15 points) A 5×5 matrix A has characteristic polynomial $-\lambda^3(\lambda - 1)(\lambda - 3)$.

(a). What values can $\dim \operatorname{Nul} A$ have?

(b). For each value n you gave in part (a), answer the following question:

5. (22 points) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 7 \end{bmatrix}$.

(a). Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

(b). Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Find the vector in V closest to $\vec{v}_3$.

(c). Find the distance between V and $\vec{v}_3$.

6. (15 points) Use methods from Math 54 to find an upper bound for the integral

$$\int_0^{\pi/2} \sqrt{x \sin x}\, dx \ .$$

Your answer may be an algebraic formula involving π and square roots, but not involving integrals, limits, or infinite sums.

[**Hint:** You may recall facts about integrals from homework problems and examples in the book.]