Math 54 Professor K. A. Ribet

Final Exam May 18, 2007

This was a 3-hour exam, 12:30–3:30PM. There were 60 points for eight questions. Problem 1 consisted of 11 true–false questions, each worth 1 point. Problems 2–8 had point values 6, 8, 9, 6, 6, 8 and6.

I think that it was a successful exam: people seemed happy, and there weren't problems that were obviously ambiguous. The only misprint that I know about was in the last problem, where "diagonal" should replace "diagonalizable." We announced this in the exam room.

**1.** *For each statement below, write TRUE or FALSE to the left of the statement. You are not required to justify your reasoning:*

*If $A$ is a square invertible matrix, then $A$ and $A^{-1}$ have the same rank.*

*If $A$ is an $m \times n$ matrix and if $b$ is in $\mathbf{R}^m$, there is a unique $x \in \mathbf{R}^n$ for which $\|Ax - b\|$ is smallest.*

*If $A$ is an $n \times n$ matrix, and if $v$ and $w$ in $\mathbf{R}^n$ satisfy $Av = 2v$, $Aw = 3w$, then $v \cdot w = 0$.*

*If the dimensions of the null spaces of a matrix and its transpose are equal, then the matrix is square.*

*If $A$ is a $2 \times 2$ matrix, then $-1$ cannot be an eigenvalue of $A^2$.*

*I liked the linear algebra portion of this course more than the differential equations portion.*

*If four linearly independent vectors lie in* $\text{Span}(\{ w_1, \ldots, w_t \})$, *then* $t$ *must be at least 4.*

*If* $B$ *is invertible, then the column spaces of* $A$ *and* $AB$ *are equal.*

*If* $A$ *is a matrix, the row spaces of* $A$ *and* $A^T A$ *are equal.*

*If two symmetric* $n \times n$ *matrices* $A$ *and* $B$ *have the same eigenvalues, then* $A = B$.

*If the characteristic polynomial of* $A$ *is* $(\lambda - 1)(\lambda + 1)(\lambda - 3)^2$, *then* $A$ *is necessarily diagonalizable.*

**2.** *Consider the vectors* $v_1 = [0, 1, 0, 1, 0]$, $v_2 = [0, 1, 1, 0, 0]$, $v_3 = [0, 1, 0, 1, 1]$ *in* $\mathbf{R}^5$. *Find* $w_1, w_2, w_3$ *in* $\mathbf{R}^5$ *such that* $w_i \cdot w_j = 0$ *for* $i \neq j$ *(*$i$ *and* $j$ *between 1 and 3), and such that* $\text{Span}(\{ w_1, \cdots, w_i \}) = \text{Span}(\{ v_1, \cdots, v_i \})$ *for* $i = 1, 2, 3$.

**3.** *Find* $x_1(t)$ *and* $x_2(t)$ *such that*

$$x_1'(t) = -2x_1(t) + 2x_2(t) \qquad x_2'(t) = +2x_1(t) + x_2(t)$$

and $x_1(0) = -1$, $x_2(0) = 3$.

**4.** *Let $A$ be the matrix* $\begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 1 & 1 & 0 \\ 4 & 2 & 4 & 2 \end{bmatrix}$. *Find bases for each of the following: the null space of $A$; the row space of $A$; the column space of $A$.*

**5.** *The theory of Fourier series implies that there are numbers $a_0, a_1, a_2, \ldots$ such that*

$$|\sin x| = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx$$

*for all real numbers $x$. Find $a_0$, $a_1$, $a_2$ and $a_3$. (It may be helpful to recall the formula $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ from trigonometry.)*

**6.** *Find $u(x,t)$ that satisfies the equation $25u_{xx} = u_t$ on the region $0 < x < \pi$, $t > 0$ as well as the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$ and $u(x,0) = \sin 3x - \sin 4x$ for $0 \leq x \leq \pi$.*

**7.** *Suppose that $v_1, \ldots, v_n$ are vectors in $\mathbf{R}^n$ and that $A$ is an $n \times n$ matrix. If $Av_1, \ldots, Av_n$ form a basis of $\mathbf{R}^n$, show that $v_1, \ldots, v_n$ form a basis of $\mathbf{R}^n$ and that $A$ is invertible.*

**8.** *Let* $v_1 = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$. *Suppose that* $A$ *is the* $3 \times 3$ *matrix for which* $Av_1 = v_1$, $Av_2 = 0$, $Av_3 = 5v_3$. *Find an invertible matrix* $S$ *and a diagonalizable matrix* $\Lambda$ *such that* $A = S\Lambda S^{-1}$.