Math 54, Summer '99
Midterm Exam

Time allowed: 1 hour.

Instructor : Malabika Pramanik.

Name: ________________

Student ID number: ________________

INSTRUCTIONS

1. **PLEASE DO NOT TURN THIS PAGE OVER UNTIL INSTRUCTED TO DO SO.**
2. Fill in your name and other details.
3. Please show your work. Solutions showing only the final answer without the intermediate steps will receive no credit. You may use the reverse side of each page for rough work.
4. The figure in brackets following each question (or part of question) denotes the number of points alloted to that question (or part of question).

Problem	Max. score	Assigned score
1	20	
2	20	
3	10	
4	20	
5	8	
6	7	
7	15	
Total	100	

1. Answer "True" or "False". Give reasons for your answer to get credit. [**5 x 4 = 20 points**]

(a) The set $\{1, x, x^2, x^3\}$ is an orthonormal basis for P_3 with inner product

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3, \quad \text{where,}$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \text{ and, } q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3.$$

(b) The set $\{1, x, x^2, x^3\}$ is an orthonormal basis for P_3 with inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx.$$

(c) If A and B are diagonal nxn matrices, then $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$.

(d) If A is an orthogonal matrix and $\mathbf{x} \in \mathbb{R}^n$, then $||\mathbf{Ax}|| = ||\mathbf{x}||$.

(e) The functions $f_1(x) = x^3$ and $f_2(x) = x^2|x|$ form a linearly independednt set on $(-\infty, \infty)$.

2. (a) [**5 points**] Find a basis for the subspace

$$W = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;:\; x_1 + x_2 - 3x_3 = 0 \right\}$$

(b) [**15 points**] Let $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$. Find $\mathbf{w}^* \in W$ such that,

$$\|\mathbf{v} - \mathbf{w}^*\| \leq \|\mathbf{v} - \mathbf{w}\|$$

for all $\mathbf{w} \in W$.

3. (a) [**5 points**] Does the formula

$$p \cdot q = p(0)q(0) + p(1)q(1)$$

define an inner product on P_2 ? Explain.

(b) [**5 points**] Answer the same question as in (a) when

$$p \cdot q = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

4. [**20 points**] For the 4x4 matrix $\mathbf{A}$ given below, find an orthogonal matrix $\mathbf{Q}$, and a diagonal matrix $\mathbf{\Lambda}$, such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{\Lambda}$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

5. [**8 points**] A matrix $\mathbf{A}$ is called *nilpotent* if $\mathbf{A}^k = \mathbf{0}$ for some positive integrer k. Show that a nonzero nilpotent matrix is not diagonalizable.

6. [**7 points**] Let $\mathbf{u} \in \mathbb{R}^n$ be such that $\mathbf{u}^T\mathbf{u} = 1$. Let $\mathbf{A}$ denote the nxn matrix

$$\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T.$$

Prove that $\mathbf{u}$ is an eigenvector of $\mathbf{A}$. What is the associated eigenvalue ?

7. **[15 points]** Compute the Wronskian of two solutions of the following differential equations *without* solving the equation.

$$t^2y'' - t(t+2)y' + (t+2)y = 0$$