Mathematics 54W

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Spring, 2005

Midterm —April 07, 2005

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing. Note that there are problems on the back of this sheet, for a total of five problems.

1. (5 pts) Let $A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$, let W be the column space of A, and let $Y := \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

   (a) (5 pts) Use the Gram-Schmidt process to find an orthogonal basis for W.

   (b) (5 pts) Find the orthogonal projection Y' of Y on W.

   (c) (5 pts) Find the distance from Y to W.

   (d) (5 pts) Find X such that $A^T A X = A^T Y$. (Hint: use part (a) to save some work.)

2. Let $A := \begin{pmatrix} 54 & 81 \\ -9 & 0 \end{pmatrix}$.

   (a) (2 pts) Find the characteristic polynomial of A.

   (b) (3 pts) Find the eigenvalues of A.

   (c) (5 pts) Find a diagonal matrix D and a nilpotent matrix N such that $A = D + N$.

   (d) (5 pts) Use part (c) to find a matrix B such that that $B^3 = A$.

3. Let $A := \begin{pmatrix} 5 & 12 \\ -2 & -5 \end{pmatrix}$.

(a) (2 pts) Find the characteristic polynomial of A.

(b) (3 pts) Find the eigenvalues of A.

(c) (5 pts) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

(d) (2 pts) List all possibilities for D. Explain.

(e) (3 pts) Find a matrix B such that $B^3 = A$.

4. Let A by an $n \times n$ matrix whose only eigenvalue is 0.

   (a) (5 pts) Is A necessarily 0? Give a proof (explanation) or counterexample. You may use a theorem proved in class.

   (b) (10 pts) Answer the same question, assuming now that A is symmetric.

5. (2 pts) Let P_2 be the vector space of polynomials of degree less than or equal to 2, and let $\mathcal{B} := (1, x, x^2)$ denote the standard ordered basis for P_2. Let $T : P_2 \to P_2$ denote the mapping sending f to $f' + f$.

   (a) (5 pts) Show that T is a linear transformatiion.

   (b) (5 pts) Find the matrix A for T with respect to the basis $\mathcal{B}$.

   (c) (5 pts) Find the eigenvectors and eigenvalues of A.

   (d) (5 pts) Is A diagonalizable? Explain.