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1. Suppose that

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

(a) (5 pts) Find the eigenvalues of A . List them with the smallest one first.

(b) (5 pts) Find corresponding eigenvectors.

(c) (10 pts) Find the limit as $n \rightarrow \infty$ of the sequence of matrices $A^n/2^n$.

2. Below are two examples of matrices A . In each case, find whether A is diagonalizable. If A is diagonalizable, find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$. If A is not diagonalizable, explain clearly how you know that it isn't.

(a) (10 pts)

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

(b) (10 pts)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

3. True or false? No justification necessary. For each question, if you answer correctly then you get two points. If you answer incorrectly then you lose two points. If you do not answer then you do not gain or lose any points. (If your total score is negative then it will be rounded up to zero.)

- T F If a square matrix has orthonormal columns then it also has orthonormal rows.
- T F If W is a subspace of \mathbb{R}^n then W and W^\perp have no vectors in common.
- T F If \mathbf{u} and \mathbf{v} are orthogonal vectors then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.
- T F An orthogonal matrix is any square matrix with orthogonal columns.
- T F Any triangular matrix is diagonalizable.
- T F If A is an $n \times n$ matrix so that $A^2 = A$ then its only possible eigenvalues are zero and one.
- T F Every invertible matrix is diagonalizable.
- T F If W is a subspace of \mathbb{R}^n then for any vector \mathbf{y} in \mathbb{R}^n , \mathbf{y} is in W^\perp if and only if $\text{proj}_W \mathbf{y} = 0$.
- T F If A is an $n \times n$ diagonalizable matrix then each vector in \mathbb{R}^n can be written as a linear combination of eigenvectors of A .
- T F If A and B are diagonalizable $n \times n$ matrices then so is $A + B$.
- T F A basis of P_n must contain a polynomial of degree j for each integer j between 0 and n .
- T F The nonzero rows of a matrix A form a basis for Row A .
- T F Row operations on a matrix cannot change the null space.

4. (a) (10 pts) Suppose that a square matrix A satisfies $(A - I)^2 = 0$. Find an explicit formula for A^{-1} . Your answer should be in terms of A , and not in terms of eigenvalues or eigenvectors.

(b) (10 pts) Find a 3×3 matrix with real entries whose eigenvalues are 1 , $2 + 3i$ and $2 - 3i$.

5. Recall that P_3 is the vector space of polynomials p of degree at most 3 in a variable t . Define a linear transformation $T : P_3 \rightarrow P_3$ by saying that for a polynomial p in P_3 , $T(p)$ is the polynomial \widehat{p} given by $\widehat{p}(t) = tp'(t)$.

a. (5 pts) What is the kernel (or null space) of T ? Justify your answer. (Hint : do not use matrices.)

b. (5 pts) Is T onto? (Equivalently, is the range of T all of P_3 ?) Justify your answer. (Hint : do not use matrices.)

c. (5 pts) Let W be the subspace of P_3 consisting of polynomials p so that $p(1) = 0$. Find a basis of W . (Hint : do not use matrices.)