Math 54 final, Spring 2010, John Lott

This is a closed everything exam, except for a 3x5 card with notes. Please put away all books, calculators and other portable electronic devices.

You need to justify every one of your answers. Correct answers without appropriate supporting work will be treated with great skepticism (except for problem number 9). At the conclusion, hand in your exam to your GSI. The last page of this exam is blank, so that you can use it for scratchwork.

Write your name on this exam and on any additional sheets that you hand in. If you need additional paper, get it from me.

Problem	Score
1	
2	
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4	
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9	
Total	

Your name ______________________________

Your GSI ______________________________

Discussion time ______________________________

Your SID ______________________________

1. (20 pts) Apply the Gram-Schmidt procedure to the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to generate an orthogonal set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

2. It is the year 2050. California has seceded from the United States to form a new country Mellownia. In order to keep good vibrations in and bad vibrations out, Mellownia has sealed its borders. The citizens of Mellownia live in two cities, Berkeley and Lodi.

The natural growth rate of Berkeley, due to births and deaths, is 2% per year.
The natural growth rate of Lodi, due to births and deaths, is 8% per year.
Every year, 1% of the Berkeley population moves to Lodi.
Every year, 6% of the Lodi population moves to Berkeley.

a. (10 pts) Let $B(t)$ denote the Berkeley population at time t and let $L(t)$ denote the Lodi population at time t. Suppose that we assemble these into a vector $\mathbf{x}(t) = \begin{bmatrix} B(t) \\ L(t) \end{bmatrix}$. Find a 2×2 matrix A so that

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

b. (10 pts) As time goes on, the populations of both Berkeley and Lodi increase. Find the limiting value of their ratio. That is, find $\lim_{t\to\infty} \frac{B(t)}{L(t)}$.

3. a. (10 pts) Find the general solution to the equation $y''(x) + y(x) = 2e^{-x}$.

b. (10 pts) Find the solution with initial values $y(0) = y'(0) = 0$.

4. (Note : you can do part b of this problem without doing part a.)

If A is a square matrix then the *trace* of A, denoted $\text{tr}(A)$, is the sum of its diagonal entries.

Fact : If A and B are $n \times n$ matrices then $\text{tr}(AB) = \text{tr}(BA)$.

a. (10 pts) Using the fact above, show that if S and T are similar matrices then $\text{tr}(S) = \text{tr}(T)$. Carefully explain your reasoning.

b. (10 pts) Using part a, show that if A is a diagonalizable square matrix then $\text{tr}(A)$ equals the sum of the eigenvalues of A.

5. (20 pts) Consider the partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

for $u(x, t)$, where $-\infty < x < \infty$ and $t \geq 0$. Using separation of variables, find a bunch of real-valued solutions to the PDE. Check that your solutions do satisfy the PDE.

6. a. (10 pts) Find a homogeneous third-order linear differential equation with constant real coefficients that has

$$y(x) = 3e^{-x} - \cos(2x)$$

as a solution.

b. (10 pts) What is the general solution of that differential equation?

7. (Note : you can do part b without doing part a, and part d without doing part c).

a. (5 pts) Find the Fourier sine series for the function $f(x) = 1$, defined for $0 \leq x \leq \pi$. Your answer should be in the form $f(x) \sim \sum(?) \sin(?)$.

b. (5 pts) For each x satisfying $0 \leq x \leq \pi$, determine what the sum of the Fourier sine series is.

c. (5 pts) Find the Fourier cosine series for the function $f(x) = 1$, defined for $0 \leq x \leq \pi$.

d. (5 pts) For each x satisfying $0 \leq x \leq \pi$, determine what the sum of the Fourier cosine series is.

8. a. (10 pts) Find the solution to the heat equation

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2},$$

defined for $0 \le x \le \pi$ and $t \ge 0$, with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial temperature distribution $u(x,0) = \sin(3x) - 5\sin(5x)$.

b. (10 pts) Suppose that we keep the same equation and initial condition, but we change the boundary condition to $u(0,t) = 0$ and $u(\pi,t) = 1$. For any x between 0 and π, find the limit when $t \to \infty$ of $u(x,t)$.

9. True or false? No justification necessary. Each question is worth three points. (Warning: there are multiple versions of this problem, with the questions permuted. If you copy off of your neighbor then there is a good chance that you'll get half of the answers wrong.)

T F If A is diagonalizable then so is A^2.

T F If the square matrix A fails to be invertible then 0 must be an eigenvalue of A.

T F If A is a square matrix and the characteristic polynomial of A is $(\lambda-6)^2(\lambda-7)^2$ then there exist two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $A\mathbf{v}_1 = 6\mathbf{v}_1$ and $A\mathbf{v}_2 = 6\mathbf{v}_2$.

T F If S is a linearly independent set of vectors in a vector space V then S is a basis for Span(S).

T F Every solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, with the boundary conditions $u(0,t) = u(1,t) = 0$, can be written in the form $u(x,t) = X(x)T(t)$.

T F If A and B are matrices so that $AB = I_n$ then A and B are both invertible.

T F Each eigenvector of an invertible matrix A is also an eigenvector of A^{-1}.

T F If $\{y_1(t), y_2(t)\}$ are linearly independent functions then the Wronskian $W[y_1, y_2](t)$ is nonzero for all t.

T F The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable.

T F If A is an orthogonal $n \times n$ matrix then $Row(A) = Col(A)$.

T F If $ay'' + by' + cy = 0$ is a constant-coefficient differential equation with $a \neq 0$ then a solution is uniquely determined by the two pieces of information $y(0)$ and $y(1)$.

T F The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

T F The least squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A which is closest to $\mathbf{b}$.