Problem 1:

a) $E_{initial} = E_{final}$

$K.E_{initial} + U_{initial} = K.E_{final} + U_{final}$

$0 + \frac{1}{2}kx^2 = K.E_{final} + mg(2r)$

$\Rightarrow K.E_{top} \equiv \boxed{K.E_{final} = \frac{1}{2}kx^2 - mg(2r)}$

b) The sign has to be positive since the spring exerts a force in the direction of motion on the mass. Therefore, $\theta = 0$ & so $\cos\theta = 1 > 0$. Hence, the dot product of $F_{spring}$ & $d\vec{x}$ is going to be positive at all time.

To find the value of $W_{spring}$ we use:

$W_{net} = \Delta K.E$

$\Rightarrow W_{spring} = -\Delta U$ (Note that $\Delta K.E = -\Delta U$ since the energy is conserved & all the Potential energy transforms into kinetic energy)

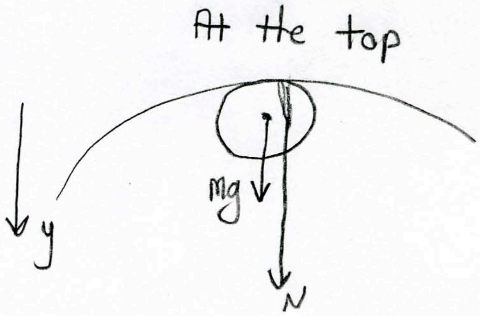
$\Rightarrow W_{spring} = -(0 - \frac{1}{2}kx^2)$

$\Rightarrow \boxed{W_{spring} = \frac{1}{2}kx^2}$

c) $\sum F_y = mg + N = ma_y$

we are given $N = 2mg$. So,

$$3mg = ma_y$$




where $a_y$ is actually the centripical acceleration, $a_c = \frac{V_{top}^2}{r}$.

So, we get:

$$3mg = m\frac{V_{top}^2}{r} \Rightarrow \boxed{V_{top} = \sqrt{3gr}}$$

d) From part c, we find: $K.E_{top} = \frac{1}{2}mV_{top}^2 = \frac{3}{2}mgr$.

From part a, we have $K.E_{top} = \frac{1}{2}kx^2 - mg(2r)$

Equate these two & Solve for x:

$$\frac{3}{2}mgr = \frac{1}{2}kx^2 - mg(2r)$$

$$\Rightarrow x^2 = \frac{7mgr}{k} \Rightarrow \boxed{x = \sqrt{\frac{7mgr}{k}}}$$

e) Static friction exists when two objects do not move relative to each other. Therefore, the work done by static friction is zero since there is no displacement. So, No need for modification in the Energy Conservation equation.

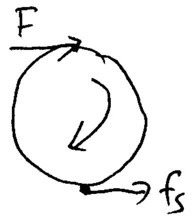
f.) Now, we have to include the rotational kinetic energy. So,

$$E_{initial} = E_{final}$$

$$\frac{1}{2}kx^2 = mg(2r) + \frac{1}{2}mv_{top}^2 + \frac{1}{2}I\omega^2$$

# Problem 2

FBD (forces only in x-direction are shown)




$$\begin{cases} F_{net,x} = F + f_s = Ma \\ \tau_{net} = FR - f_s R = I\alpha \end{cases}$$

rolling w/o slipping $a = R\alpha$

$$I\alpha = \frac{1}{2}MR^2\alpha = \frac{1}{2}MR(R\alpha) = \frac{1}{2}MRa$$

$$\begin{cases} F + f_s = Ma \\ F - f_s = \frac{1}{2}Ma \end{cases}$$

a)

$$\begin{array}{rl} & F + f_s = Ma \\ + & F - f_s = \frac{1}{2}Ma \\ \hline & 2F = \frac{3}{2}Ma \end{array}$$

$a = \frac{4}{3}\frac{F}{M}$ to the right

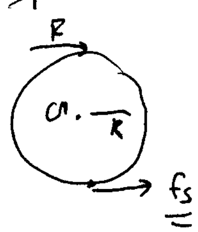
b) $\alpha = \frac{a}{R} = \frac{4}{3}\frac{F}{RM}$ into the page

c)

$$\begin{array}{rl} & F + f_s = Ma \\ - & F - f_s = \frac{1}{2}Ma \\ \hline & 2f_s = \frac{1}{2}Ma \end{array}$$

$f_s = \frac{1}{4}Ma = \frac{1}{\cancel{4}}\cancel{M}\frac{\cancel{4}}{3}\frac{F}{\cancel{M}} = \frac{F}{3}$ to the right

diagram




d) $K.E. = \frac{1}{2}Mv_f^2 + \frac{1}{2}I\omega_f^2$

rolling w/o slipping $\quad v_f = R\omega_f$

$$I = \frac{MR^2}{2}$$

$$\Rightarrow \frac{1}{2} I \omega_f^2 = \frac{1}{2}\left(\frac{MR^2}{2}\right)\omega_f^2 = \frac{1}{4} M (R\omega_f)^2 = \frac{1}{4} M v_f^2$$

so $K.E. = \frac{1}{2} M v_f^2 + \frac{1}{2} I \omega_f^2 = \frac{1}{2} M v_f^2 + \frac{1}{4} M v_f^2 = \frac{3}{4} M v_f^2$

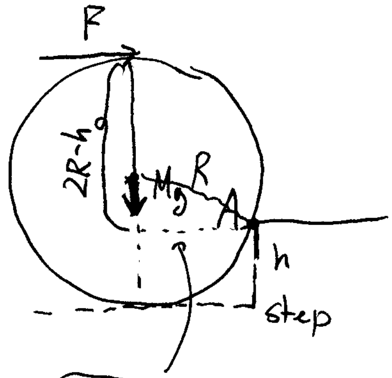
since $F$ is constant, $a$ is constant

$$v_f^2 = \cancelto{0}{v_0^2} + 2aL$$

$$v_f^2 = 2aL = \frac{8}{3}\frac{F}{M}L$$

so $K.E = \frac{3}{4} M v_f^2 = \frac{\cancel{3}}{\cancel{4}} \cancel{M} \frac{\cancel{8}}{\cancel{3}} \frac{F}{\cancel{M}} L = 2FL$

e)




$\sqrt{R^2 - (R-h)^2} = \sqrt{h(2R-h)}$

new axis of rotation is through pt. A

in order for the spool to climb the step, we need to rotate it about point A.

$F$, $Mg$ are two forces that give rise to the torque in this rotation.

$$\tau_{net} = F R_F - Mg R_{Mg}$$

lever arm corresponding to force $F$ is $R_F = 2R - h$

lever arm corresponding to force $Mg$ is $R_{Mg} = \sqrt{h(2R-h)}$

need $\tau_{net} = F(2R-h) - Mg\sqrt{h(2R-h)} \geq 0$ in order to accomplish rotation about point A

$$F_{min} = Mg\frac{\sqrt{h(2R-h)}}{2R-h} = Mg\sqrt{\frac{h}{2R-h}}$$

# $\boxed{\text{Solution of Problem 3}}$ (corrected) (30 points)

$V^{(1)}_{initial} = 3\,m/s$ $\quad V^{(2)}_{initial} = 0\,m/s$ $\quad V^{(3)}_{initial} = 1\,m/s$ $\qquad m_1 = m_2 = m_3 = m = 1\,kg$

① $\qquad$ ② $\qquad$ ③

$x = 0$ $\qquad x = 2\,m$ $\qquad x = 3\,m$

$\boxed{5\,pts}$ a)

$$\boxed{X_{CM}} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\cancel{m}(x_1 + x_2 + x_3)}{3\cancel{m}} = \frac{0 + \overset{\text{meters}}{2m} + \overset{\text{meters}}{3m}}{3} \boxed{= \frac{5}{3}m \approx 1.67\,\text{meters}}$$

$\boxed{5\,pts}$ b)

$E_{total} = \frac{mv_1^2}{2} + \frac{mv_2^2}{2} + \frac{mv_3^2}{2} = \text{const}$ in time (<u>elastic collisions</u> preserve energy)

$$\Rightarrow E_{total}(t=3\,sec) = E_{total}(t=1\,sec) = E_{total}(t=0\,sec) = \frac{m}{2}(v_1^2 + v_2^2 + v_3^2) = \frac{1\,kg}{2}\left(9\frac{m^2}{s^2} + 0^2 + 1\frac{m^2}{s^2}\right) = \boxed{5\,J}$$

$\boxed{5\,pts}$ c)

At the moment of time when 1 and 2 just stick together ($t_1 = \frac{2\,meters}{V^{(1)}_{initial}} = \frac{2}{3}\,sec$)

$$X_{CM}(t_1) = \frac{2\,meters \cdot (2\cancel{m}) + \left(3\,meters + \frac{2}{3}\,sec \cdot 1\,\frac{meter}{sec}\right)\cdot \cancel{m}}{3\cancel{m}} = \frac{23}{9}\,meters$$

↳ this term is due to displacement of the third ball after time $t_1 = \frac{2}{3}\,sec$ passes. It was missing in the earlier posted solution, but was taken into account in grading of this problem.

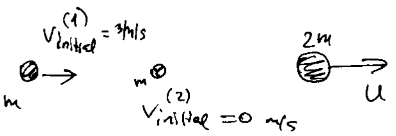
$\Downarrow$

$$\boxed{X_{CM}(t_1) = \frac{23}{9}\,meters \approx 2.56\,meters}$$

$\boxed{5\,pts}$ d)

It is clear that first collision happens after time $t_1 = \frac{2\,meters}{3\,meters/sec} = \frac{2}{3}\,sec$ passes. It is inelastic $\Rightarrow$ momentum conservation

gives:




$$\cancel{m} \cdot V^{(1)}_{initial} + \cancel{m} \cdot 0 = 2\cancel{m} \cdot u \Rightarrow \qquad u = \frac{1}{2} V^{(1)}_{initial} = \frac{3}{2}\,meters/sec = 1.5\,\frac{meters}{sec}$$

The second collision happens after time $\Delta t$ after $t_1$, s.t.

$$\underbrace{2\,meters}_{x_2} + \underbrace{u}_{1.5\frac{meters}{sec}} \cdot \Delta t = \underbrace{3\,meters}_{x_3} + \underbrace{\frac{2}{3}\,sec \cdot 1\,\frac{meter}{sec}}_{\text{displ. term, see above}} + 1\,\frac{meter}{sec} \cdot \Delta t \Rightarrow$$

$$\Rightarrow 0.5\,\Delta t = \frac{5}{3}\,meters \Rightarrow \Delta t = \frac{10}{3}\,sec \Rightarrow \qquad t_2 = t_1 + \Delta t = \frac{2}{3}\,sec + \frac{10}{3}\,sec = 4\,sec$$

So, the second collision happens after the 4 sec since the initial $t=0$ moment of time. The first collision as we already know happens at $\frac{2}{3}$ sec.

(Problem 3) (continued...)

Thus at ~~a~~ time = 1 sec and time = 3 sec only first collision has happened but not yet the second one : $t_1 < 1\,sec < t_2$, $t_1 < 3\,sec < t_2$.

At $t=0$ the total energy was found in (b): $\boxed{E_{total}(t=0) = 5\,J}$

At $t=3$ and $t=1\,sec$:

$$\boxed{E_{total}(t=3\,sec) = E_{total}(t=1\,sec) = \frac{\cancel{2}m\cdot u^2}{\cancel{2}} + \frac{mv_3^2}{2} = \left(1.5^2 + \frac{1}{2}\cdot 1^2\right) J = 2.75\,J}$$

[5pts] e). $\boxed{\text{center of mass = geometrical center = center of the middle ring}}$

(because of symmetry)

After


[5pts] f)

~~P~~ Momentum is conserved:

$$M\vec{v}_0 = \vec{P}_f = (3M)\cdot\vec{V}_{CM} \Rightarrow \boxed{\vec{V}_{CM} = \frac{\vec{V}_0}{3}}.$$

Problem 4

a) $$F = \frac{dp}{dt} = \frac{dm * v_r}{dt} = \frac{dm}{dt} v_r$$
$$= M(t) a = M(t) \frac{dv}{dt}$$
Here $v_r$ is the relative velocity between water and the cart.

Since the velocity of water is changing from $v_w$ to V(t), $v_r = v_w - v(t)$.

So $\frac{dv}{dt} = \frac{1}{M(t)} \frac{dm}{dt} v_r = \frac{1}{M(t)} \frac{dm}{dt} (v_w - v(t))$.

b) $\frac{dm}{dt} = \frac{dm}{dl} * \frac{dl}{dt} = \lambda \frac{dl}{dt}$

& $\frac{dl}{dt} = v_r$ is also the relative velocity between water and the cart.

Finally $\frac{dm}{dt} = \lambda v_r = \lambda(v_w - v(t))$.

If you find it's not so easy to understand why it should be the relative velocity, not $v_w$, just try to switch to the moving frame of the cart. Image you're moving with the cart, the velocity of the water now becomes $v_w$-v(t), then it become natural that $\lambda(v_w - v(t))$ represents the amount of water coming into collision with the cart in the time interval dt.