Second midterm of Math 54, Fall 2012

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Problem	Points Earned	Total Points
1		30
2		15
3		20
4		10
5		25
Total		100

(*30 points total*) Consider the matrix

$$A = \begin{bmatrix} 1 & a & 1 \\ 0 & -1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

1. (*10 points*) State all necessary conditions, if any, on a, b for A to be invertible. Justify your answer.

2. (*20 points*) State all necessary conditions, if any, on a, b for A to be diagonalizable. Justify your answer.

1. Since A is triangular, $\det A = 1(-1)1 = -1 \neq 0$, so A is invertible for $\boxed{\text{all } a, b}$.

2. Since A is triangular, the eigenvalues of A are 1 and -1 with algebraic multiplicities 2 and 1.

A will be diagonalizable if and only if the geometric multiplicities of 1 and -1 are 2 and 1. But since

$1 \leq$ geom. mult. $\leq$ alg. mult,

the geometric multiplicity of -1 is always 1, so we only need to find the geometric multiplicity of 1, ie. $\dim \text{Nul}(A - I)$.

$$A - I = \begin{bmatrix} 0 & a & 1 \\ 0 & -2 & b \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & -2 & b \\ 0 & a & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 + \frac{a}{2} R_1} \begin{bmatrix} 0 & -2 & b \\ 0 & 0 & 1 + \frac{ab}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

$\dim \text{Nul}(A - I) = 3 - (\dim \text{Col}(A - I)) = 3 - (\# \text{ of pivots of } A - I)$, which is 2 only when $1 + \frac{ab}{2} = 0$.

So $\boxed{A \text{ is diagonalizable if and only if } 1 + \frac{ab}{2} = 0}$

ie. $\boxed{ab = -2}$

<u>WARNING</u>: Answers like "$a = -\frac{2}{b}$" are not completely correct; a and/or b may be 0, so it does not make sense to divide by them.

(*15 points*) Find the closest point to $\mathbf{y}$ in the subspace W of $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution #1: $u_1 \cdot u_2 = 0+1+2 = 3 \neq 0$, so u_1 and u_2 are not orthogonal, so do Gram-Schmidt to get an orthogonal basis of W:

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 = u_2 - \frac{0+1+2}{1+1+1} v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Then the closest point is

$$\hat{y} = \mathrm{proj}_W \vec{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{-1+3+1}{1+1+1} v_1 + \frac{1+0+1}{1+0+1} v_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

Warning: The answer is not:

$$\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{-1+3+1}{1+1+1} u_1 + \frac{0+3+2}{0+1+4} u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

First, $\left\| \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right\| = \sqrt{4+1+4} = 3$ is bigger than:

$\left\| \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\| = \sqrt{1+4+1} = \sqrt{6}$.

Second, $\vec{y} - \hat{y}$ needs to be orthogonal to both $\vec{u}_1$ and $\vec{u}_2$;

$\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ is not orthogonal to $\vec{u}_1$.

In general, the projection formula does not give the right answer if the starting vectors are not orthogonal.

Solution #2: Want $c_1\vec{u}_1 + c_2\vec{u}_2$ where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is the least-squares solution to $\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}}_{\vec{b}}$

So compute: $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$. $A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 5 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 3 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{3}{2}R_2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$ So $c_1 = 0$, $c_2 = 1$,

final answer is $0\vec{u}_1 + 1\vec{u}_2 = \boxed{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$

(*20 points*) Find any orthonormal basis for the inner product on $\mathbb{P}_2(x)$ - i.e. polynomials in x of degree up to 2 - given by

$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx.$$

A basis for $\mathbb{P}_2$ is $\{\underset{u_0}{1}, \underset{u_1}{x}, \underset{u_2}{x^2}\}$. We do Gram-Schmidt to get

an orthogonal basis:

$V_0 = u_0 = 1$

$$u_1 - \frac{\langle u_1, V_0\rangle}{\langle V_0, V_0\rangle} V_0 = u_1 - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx} V_0 = u_1 - \frac{\left[\frac{x^2}{2}\right]_{x=0}^1}{[x]_{x=0}^1} V_0 = x - \frac{1}{2}.$$

Rescale: $V_1 = 2x - 1$

$$u_2 - \frac{\langle u_2, V_0\rangle}{\langle V_0, V_0\rangle} V_0 - \frac{\langle u_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1 = u_2 - \frac{\int_0^1 x^2\,dx}{\int_0^1 1\,dx} V_0 - \frac{\int_0^1 (2x^3 - x^2)\,dx}{\int_0^1 (4x^2 - 4x + 1)\,dx}$$

$$= u_2 - \frac{\left[\frac{x^3}{3}\right]_{x=0}^1}{[x]_{x=0}^1} V_0 - \frac{\left[\frac{x^4}{2} - \frac{x^3}{3}\right]_{x=0}^1}{\left[\frac{4}{3}x^3 - 2x^2 + x\right]_{x=0}^1} V_1 = u_2 - \frac{1}{3}V_0 - \frac{\frac{1}{6}}{\frac{1}{3}} V_1 = x^2 - \frac{1}{3} - \frac{1}{2}(2x-1) = x^2 - x + \frac{1}{6}$$

Rescale: $V_2 = 6x^2 - 6x + 1$

$$\|V_0\| = \sqrt{\int_0^1 1^2\,dx} = \sqrt{[x]_{x=0}^1} = \sqrt{1} = 1$$

$$\|V_1\| = \sqrt{\int_0^1 (2x-1)^2\,dx} = \sqrt{\int_0^1 (4x^2 - 4x + 1)\,dx} = \sqrt{\left[\frac{4}{3}x^3 - 2x^2 + x\right]_{x=0}^1} = \sqrt{\frac{4}{3} - 2 + 1} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\|V_2\| = \sqrt{\int_0^1 (6x^2 - 6x + 1)^2\,dx} = \sqrt{\int_0^1 (36x^4 + 36x^2 + 1 - 72x^3 + 12x^2 - 12x)\,dx} = \sqrt{\left[\frac{36}{5}x^5 - 18x^4 + 16x^3 - 6x^2 + x\right]_{x=0}^1}$$

$$= \sqrt{\frac{36}{5} - 18 + 16 - 6 + 1} = \sqrt{\frac{36}{5} - 7} = \sqrt{\frac{36}{5} - \frac{35}{5}} = \frac{1}{\sqrt{5}}.$$

So an orthonormal basis is $\boxed{\{1, \sqrt{3}(2x-1), \sqrt{5}(6x^2 - 6x + 1)\}}$.

(There are others. For example, if you started with $\{x^2, x, 1\}$, you will get $\{\sqrt{5}x^2, \sqrt{3}(5x^2 - 4x), 10x^2 - 12x + 3\}$.)

(*10 points*) Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a useful expression for the matrix for A^k, where k is an arbitrary integer > 0. Hint: Use eigenvalues and eigenvectors.

Char. poly. of $A = \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = (7-\lambda)(1-\lambda) + 8 = \lambda^2 - 8\lambda + 7 + 8$

$= \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5).$

E-vals of A = roots of Char. poly. of A = 3, 5.

$A - 3I = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, so $Nul(A-3I) = Span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ e-vect for $\lambda = 3$

$A - 5I = \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $Nul(A-5I) = Span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ e-vect for $\lambda = 5$.

So if $P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, then P is invertible and $A = P\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}P^{-1}$

So $A^k = P\begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix}P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix}\left(\frac{1}{-1+2}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3^k & 5^k \\ -2\cdot 3^k & -5^k \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -3^k + 2\cdot 5^k & -3^k + 5^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix}}$$

(*25 points total*)

1. (*15 points*) Find the equation $y = \alpha_0 + \alpha_1 x$ of the least square line that best fits the data points $(2,1), (5,2), (7,3), (8,3)$.

2. (*10 points*) For the same data points consider the equation $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4$ of the best fit least square degree four polynomial. True or false: this has a unique solution. Justify your answer.

1. Want to find all least-squares solutions to $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

These are the exact solutions $\hat{x}$ to $A^TA\hat{x} = A^T\vec{b}$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 4+25+49+64 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$

$$\begin{array}{r} 2 \\ 64 \\ 49 \\ +29 \\ \hline 142 \end{array} \qquad \begin{array}{r} 1 \\ 49 \\ +29 \\ \hline 78 \\ +64 \\ \hline 142 \checkmark \end{array}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2+10+(7+8)\cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12+45 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}.$$

$$\det(A^TA) = 4\cdot 142 - 22\cdot 22 = 4(142 - 11\cdot 11) = 4(142-121) = 4\cdot 21 = 84,$$

so $(A^TA)^{-1}$ exists and is $\frac{1}{84}\begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} = \frac{1}{42}\begin{bmatrix} 71 & -11 \\ -11 & 2 \end{bmatrix}$.

$$\text{So } \hat{x} = (A^TA)^{-1}A^T\vec{b}) = \frac{1}{42}\begin{bmatrix} 71 & -11 \\ -11 & 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{42}\begin{bmatrix} 71\cdot 9 - 11\cdot 57 \\ -11\cdot 9 + 2\cdot 57 \end{bmatrix}$$

$$\text{Ans: } y = \frac{71\cdot 9 - 11\cdot 57}{42} + \frac{-11\cdot 9 + 2\cdot 57}{42}x = \frac{71\cdot 3 - 11\cdot 19}{14} + \frac{-11\cdot 3 + 2\cdot 19}{14}x$$

$$= \frac{213 - 209}{14} + \frac{-33+38}{14}x = \boxed{\frac{2}{7} + \frac{5}{14}x}$$

2. We need to find all least-squares solutions to $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \\ 1 & 7 & 7^2 & 7^3 & 7^4 \\ 1 & 8 & 8^2 & 8^3 & 8^4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

But A has more columns than rows, so cols of A are dependent, so $\boxed{\text{FALSE}}$; there will be infinitely many solutions. (See Thm. 14 from Ch. 6.)