1. (15 points) Formulate the following problem as a linear program. Your inequalities should be preceded by a definition of the variables in clear concise English.

Suppose that a farmer has a piece of farm land, say L km^2, to be planted with either wheat or barley or some combination of the two. The farmer has a limited amount of fertilizer, F kilograms, and insecticide, P kilograms. Every square kilometer of wheat requires F_1 kilograms of fertilizer, and P_1 kilograms of insecticide, while every square kilometer of barley requires F_2 kilograms of fertilizer, and P_2 kilograms of insecticide. Let S_1 be the selling price of wheat per square kilometer, and S_2 be the price of barley. How many square kilometers of the area of land should the farmer plant with wheat and barley in order to maximize the profit?

Solution:

If we denote the area of land planted with wheat and barley by x_1 and x_2 respectively, this problem can be formulated as:

max	$S_1x_1 + S_2x_2 - F_1x_1 - F_2x_2 - P_1x_1 - P_2x_2$	(maximize the profit)
s.t.	$0 \le x_1 + x_2 \le L$	(limit on total area)
	$0 \le F_1x_1 + F_2x_2 \le F$	(limit on fertilizer)
	$0 \le P_1x_1 + P_2x_2 \le P$	(limit on insecticide)
	$x_1 \ge 0, x_2 \ge 0$	(cannot plant a negative area).

2. (15 points) Formulate the *Knapsack Problem*.

Solution:

We have n kinds of items, 1 through n. Each kind of item i has a value v_i and a weight w_i. Assume that all values and weights are nonnegative. The maximum weight that we can carry in the bag is W.

Let x_i be the decision variable whether the i^{th} item is chosen ($x_i = 1$) or not ($x_i = 0$). The knapsack problem can be formulated as:

max $\sum_{i=1}^{n} v_i x_i$

s.t. $\sum_{i=1}^{n} w_i x_i \leqslant W, \qquad x_i \in \{0,1\}$

3. (50 points) Consider the following linear programming problem P1:

$$
\text{P1:}\quad
\begin{aligned}
\max \quad & J = -x_1 - 4x_2 \\
s.t. \quad & x_2 \geq x_1 + 1 \\
& x_2 \geq -x_1 + 3 \\
& x_2 \leq 6 \\
& x_2 \geq 0 \\
& x_1 \leq 10
\end{aligned}
$$

3.1. (10 points) Solve this problem graphically in Figure 1. Give the optimal cost as well as the point (x_1,x_2) at which it is obtained.

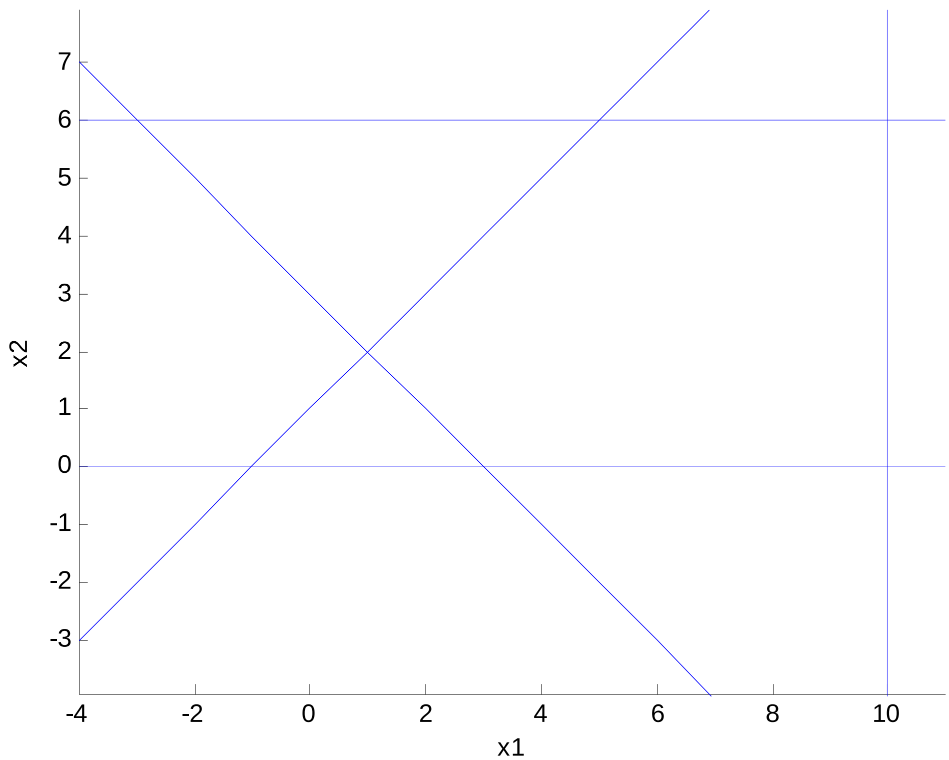


Figure 1

(Please look at “MidtermSolu3.1”, found below)

3.2. (10 points) Write down all inactive (non-binding) constraints of this problem.

$$x_2 \le 6$$
$$x_2 \ge 0$$
$$x_1 \le 10$$

3.3. (10 points) Consider the following programming problem P2:

P2:

$$\begin{aligned} \max \quad & J = -x_1 - 4x_2 \\ s.t. \quad & x_2 \ge x_1 + 1 \\ & x_2 \ge -x_1 + 3 \\ & x_2 \le 6 \\ & x_2 \ge 0 \\ & x_1 \le 10 \\ & x_1 \le 0 \end{aligned} \qquad or \qquad \begin{aligned} & x_2 \ge x_1 + 1 \\ & x_2 \ge -x_1 + 3 \\ & x_2 \le 6 \\ & x_2 \ge 0 \\ & x_1 \le 10 \\ & x_1 \ge 2 \end{aligned}$$

Rewrite problem P2 into a mixed integer linear programming problem (MILP) P3.

Solution:

P3:

$$\begin{aligned} \max \quad & J = -x_1 - 4x_2 \\ s.t. \quad & x_2 \ge x_1 + 1 \\ & x_2 \ge -x_1 + 3 \\ & x_2 \le 6 \\ & x_2 \ge 0 \\ & x_1 \le 10 \\ & x_1 \ge 2 - Md \\ & x_1 \le 0 + M(1 - d) \end{aligned}$$

where $M \gg 1$ is a large number, and $d \in \{0,1\}$ is an integer decision variable.

3.4. (10 points) Shade in Figure 2 the feasible set of the MILP P3 in 3.3:

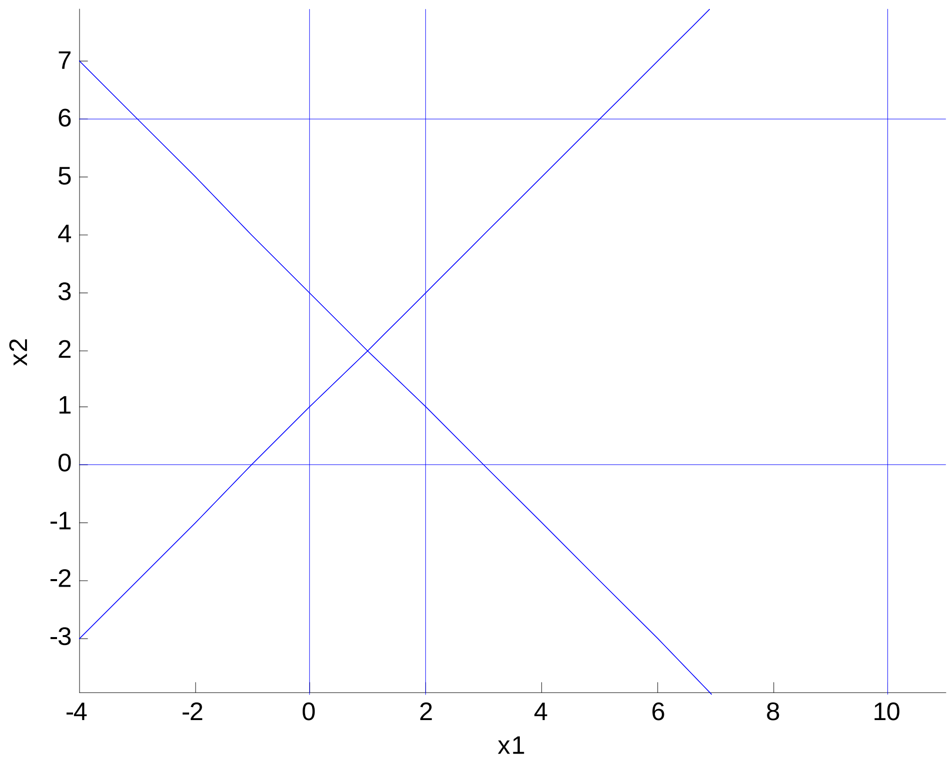


Figure 2

(Please look at “MidtermSolu3.4,” found below)

3.5. (10 points) Solve the MILP P3 in 3.3. Explain your reasoning, you can illustrate on Figure 3.

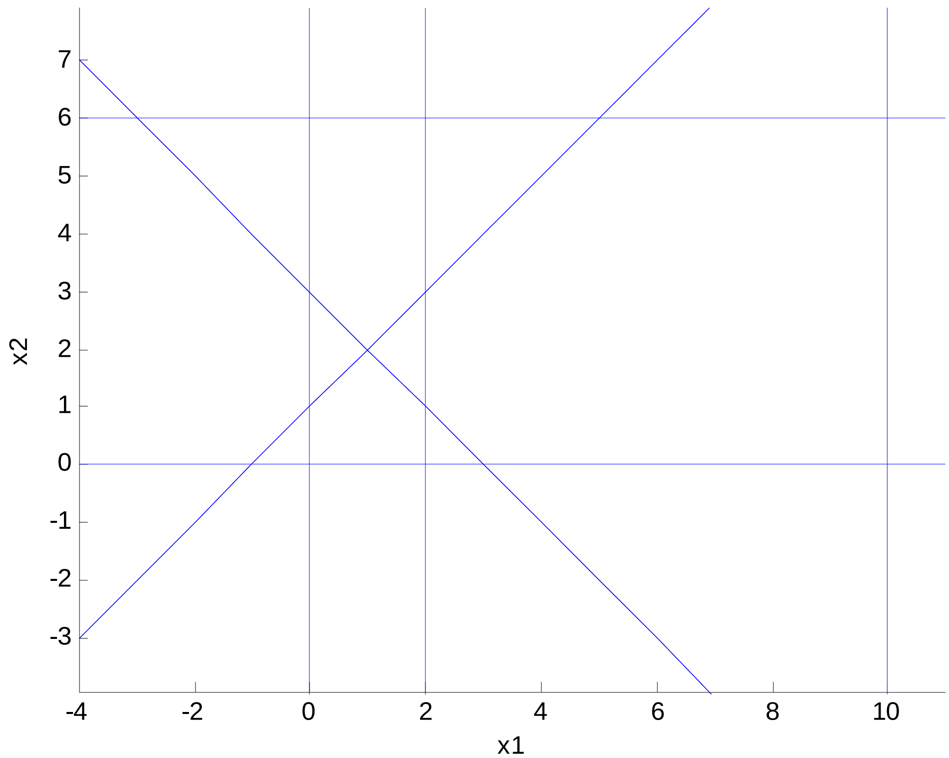


Figure 3

(Please look at “MidtermSolu3.5,” found below)

4. (20 points) Solve the following problem:

$$\max \quad (x-3)^2 + (y-2)^2$$
$$s.t. \quad \left\langle \begin{bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{bmatrix}, \begin{bmatrix} x-3 \\ y-2 \end{bmatrix} \right\rangle \le \cos\frac{\theta}{2}$$

where k = 0, 1, 2, ..., 7, and $\theta = \frac{\pi}{4}$, and $\langle . , . \rangle$ denotes inner product.

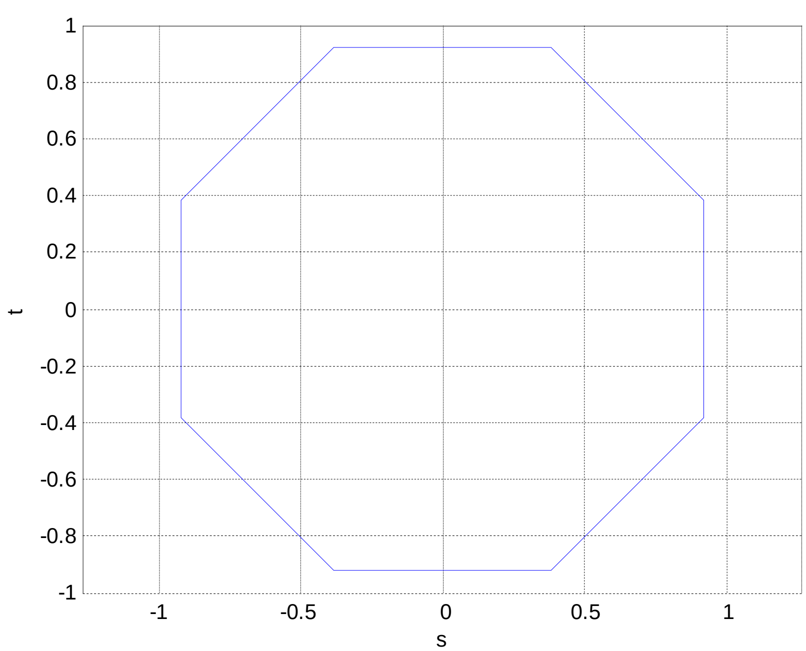
Solution:

Change of variables:
Let s = x-3, t = y-2, we have

$$\max \quad s^2 + t^2$$
$$s.t. \quad \left\langle \begin{bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{bmatrix}, \begin{bmatrix} s \\ t \end{bmatrix} \right\rangle \le \cos\frac{\theta}{2}$$

where k = 0, 1, 2, ..., 7, and $\theta = \frac{\pi}{4}$

Notice that feasible region from the eight constrains (k = 0, 1, 2, ..., 7) is a regular octagon, and the maximum of the objective function is the square of the radius of the regular octagon $r^2 = 1$, which is obtained in any of the eight vertices of the octagon $(s,t) = \left(\cos\frac{(2k+1)\theta}{2}, \sin\frac{(2k+1)\theta}{2}\right)$, where k = 0, 1, 2, ..., 7.



Change variable back, so the maximum is 1, which is obtained at
$\left(\cos\frac{(2k+1)\theta}{2}+3, \sin\frac{(2k+1)\theta}{2}+2\right)$, where k = 0, 1, 2, ..., 7, and $\theta = \frac{\pi}{4}$.
Numerically, they are
(3.9239, 2.3827)
(3.3827, 2.9239)
(2.6173, 2.9239)
(2.0761, 2.3827)
(2.0761, 1.6173)
(2.6173, 1.0761)
(3.3827, 1.0761)
(3.9239, 1.6173)

3. (50 points) Consider the following linear programming problem P1:

$$\text{P1:}\quad \begin{aligned} \max \quad & J = -x_1 - 4x_2 \\ s.t. \quad & x_2 \geq x_1 + 1 \\ & x_2 \geq -x_1 + 3 \\ & x_2 \leq 6 \\ & x_2 \geq 0 \\ & x_1 \leq 10 \end{aligned}$$

3.1. (10 points) Solve this problem graphically in Figure 1. Give the optimal cost as well as the point (x_1,x_2) at which it is obtained.

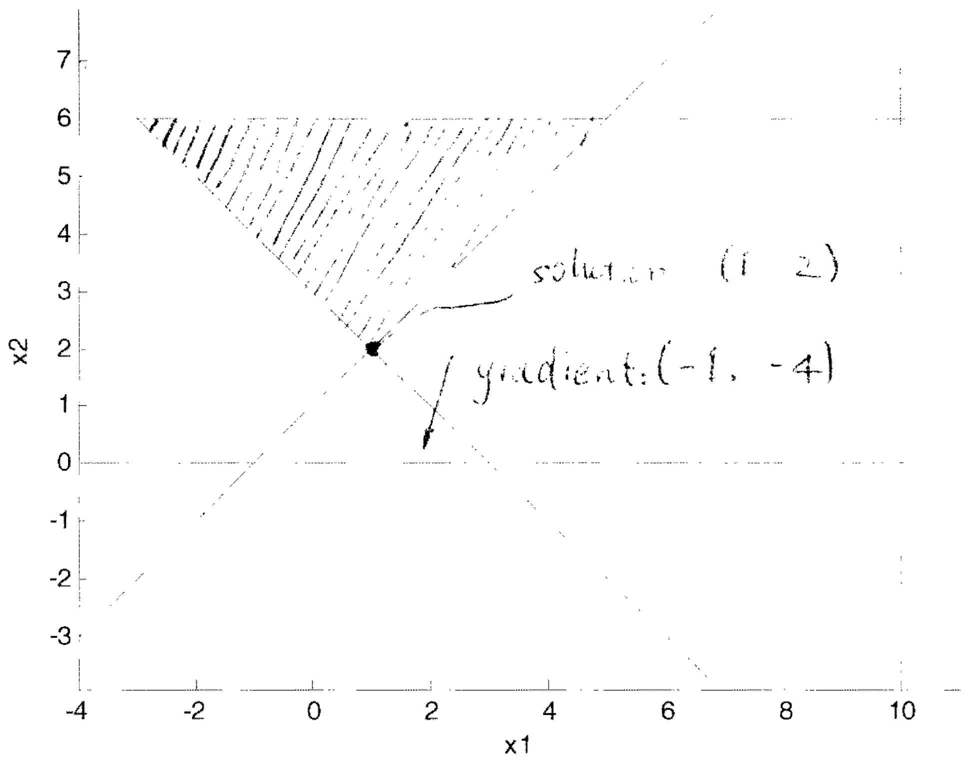


Figure 1

Solution: Optimal cost $= -1 - 4 \times 2 = -9$

It is obtained at (1, 2)

3.4. (10 points) Shade in Figure 2 the feasible set of the MILP P3 in 3.3:

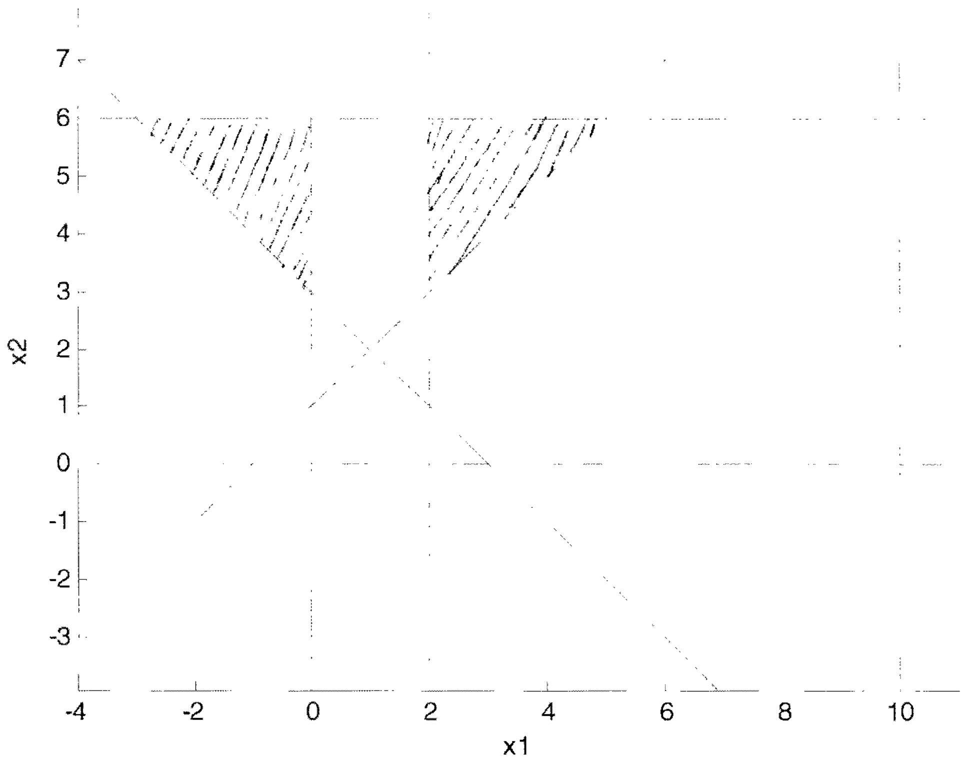


Figure 2

3.5. (10 points) Solve the MILP P3 in 3.3. Explain your reasoning, you can illustrate on Figure 3.

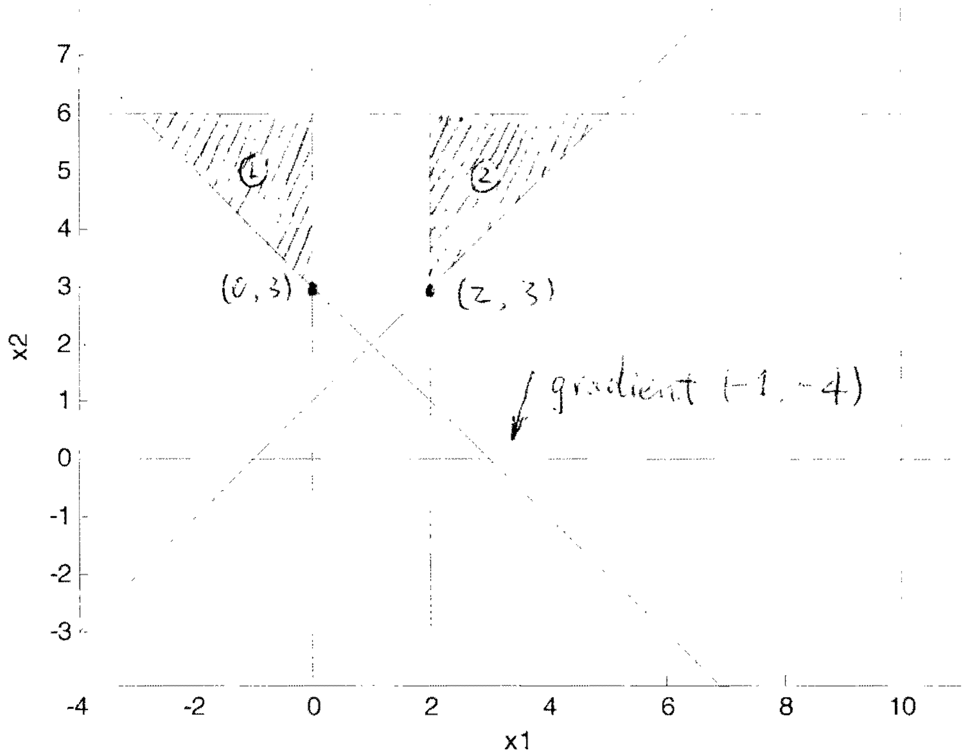


Figure 3

Solution: The MILP has two feasible regions.

Feasible region ① when $d = 1$

Optimal cost $= -12$, obtained at $(0, 3)$

Feasible region ② when $d = 0$

Optimal cost $= -14$, obtained at $(2, 3)$

So the optimal cost of the MILP P_3 is -12.

It is obtained at $x_1 = 0$, $x_2 = 3$, $d = 1$.