ME 106 Midterm I Solution

1) a) No torque in the x or z by symmetry,

So we only need the y- component of $\underline{T}$

or Because $\underline{T} = \underline{r} \times \underline{F}$, which means $\underline{T}$ is perpendicular to both $\underline{r}$ and $\underline{F}$, while $\underline{r}$ and $\underline{F}$ are on the x-z plane

So $\underline{T}$ is just in y direction.

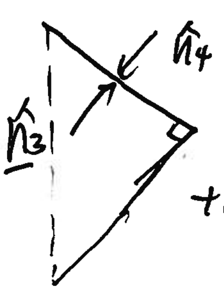
b). $\underline{r} = (x, y, z) - (0, 0, z_1) = (x, y, z - z_1)$

There is a relationship between x and z:

$$\frac{x}{z_1 - z} = \tan 30° = \frac{1}{\sqrt{3}} \qquad x = \frac{z_1 - z}{\sqrt{3}}$$

$$\underline{r} = \left(\frac{z_1 - z}{\sqrt{3}}\right)\hat{x} + y\hat{y} + (z - z_1)\hat{z}$$

c). the normal vector that goes from ρ_3 to upper section



$$\underline{n}_3 = \frac{1}{2}\hat{z} + \frac{\sqrt{3}}{2}\hat{x} = \cos 30°\hat{x} + \sin 30°\hat{z}$$

the normal vector goes from fluid ρ_4 to upper section

$$\underline{n}_4 = -\underline{n}_3 = -\cos 30°\hat{x} - \sin 30°\hat{z} = -\frac{1}{2}\hat{z} - \frac{\sqrt{3}}{2}\hat{x}$$

d) Because the partition is sealed, $P_L(z) \neq P_R(z)$

the pressure on the left side: P_L

$$P_L = P_{atm} \qquad z \geq 0$$

$$P_L = P_{atm} - \rho_3 g z \qquad \bar{z} \leq z \leq 0$$

$$P_L = P_{atm} - \rho_3 g \bar{z} - \rho_1 g (z - \bar{z}) \qquad z_B \leq z \leq \bar{z}$$

the pressure on the right side: P_R

$$P_R = P_{atm} \qquad z \geq z_4$$

$$P_R = P_{atm} - \rho_4 g (z - z_4) \qquad \bar{z} \leq z \leq z_4$$

$$P_R = P_{atm} - \rho_4 g (\bar{z} - z_4) - \rho_1 g (z - \bar{z}) \qquad z_B \leq z \leq \bar{z}$$

e) $$\underline{\Gamma} = \int dA \, \underline{r} \times [\hat{n}_3 P_L(z) + \hat{n}_4 P_R(z)]$$

$$= \int dA \, \underline{r} \times \hat{n}_3 [P_L(z) - P_R(z)]$$

$$\underline{r} = \left(\frac{z_1 - z}{\sqrt{3}}\right)\hat{x} + y\hat{y} + (z - z_1)\hat{z}$$

$$\hat{n}_3 = \frac{1}{2}\hat{z} + \frac{\sqrt{3}}{2}\hat{x}$$

$$\underline{r} \times \hat{n}_3 = \left(\frac{z_1 - z}{\sqrt{3}}\hat{x} + y\hat{y} + (z - z_1)\hat{z}\right) \times \left(\frac{1}{2}\hat{z} + \frac{\sqrt{3}}{2}\hat{x}\right)$$

$$= \frac{1}{2}\left[\frac{z - z_1}{\sqrt{3}}\hat{y} + \hat{y}\cdot\sqrt{3}(z - z_1)\right] = \frac{(z - z_1)}{2}\frac{4}{\sqrt{3}}\hat{y}$$

$$= \frac{2(z - z_1)}{\sqrt{3}}\hat{y}$$

$P_L(z) = P_{atm} - \rho_3 g z \;,\quad P_R(z) = P_{atm} - \rho_4 g (z - z_4) \qquad \bar{z} \le z \le z_4$

$P_L(z) - P_R(z) = \rho_4 g z - \rho_3 g z - \rho_4 g z_4 = g[(\rho_4 - \rho_3) z - \rho_4 z_4]$

$\underline{I} = \int dA \; \frac{2(z - z_1)}{\sqrt{3}} \hat{y} \cdot g\, [(\rho_4 - \rho_3) z - \rho_4 z_4] \,\hat{y}$

$= \frac{2g}{\sqrt{3}} \int dA \; [(\rho_4 - \rho_3) z - \rho_4 z_4] (z - z_1) \,\hat{y}$

$dA = L \cdot dz / \cos 30^\circ = dz\, L \frac{2}{\sqrt{3}}$

$\hat{y} \cdot \underline{I} = \frac{2g}{\sqrt{3}} \frac{2L}{\sqrt{3}} \int_{\bar{z}}^{z_1} dz\, (z - z_1) [(\rho_4 - \rho_3) z - \rho_4 z_4]$

$= \frac{4}{3} g L \int_{\bar{z}}^{z_1} dz \; [(\rho_4 - \rho_3) z^2 + (\rho_3 z_1 - \rho_4 z_1 - \rho_4 z_4) z$

$+ \rho_4 z_1 z_4]$

$= \frac{4}{3} g L \Big[_{\bar{z}}^{z_1} \; (\rho_4 - \rho_3) \frac{z^3}{3} + (\rho_3 z_1 - \rho_4 z_1 - \rho_4 z_4) \frac{z^2}{2} + \rho_4 z_1 z_4 z$

$= \frac{4}{3} g L \;[(\rho_4 - \rho_3) \frac{z_1^3 - \bar{z}^3}{3} + (\rho_3 z_1 - \rho_4 z_1 - \rho_4 z_4) \frac{z_1^2 - \bar{z}^2}{2} + \rho_4 z_1 z_4 (z_1 - \bar{z})]$

$= \frac{\rho_3 - \rho_4}{6} z_1^3 + \frac{\rho_3 - \rho_4}{3} \bar{z}^3 + \frac{\rho_4 - \rho_3}{2} z_1 \bar{z}^2 + \frac{\rho_4}{2} z_4 z_1^2 + \frac{\rho_4}{2} z_4 \bar{z}^2 - \rho_4 z_1 z_4 \bar{z}$

2). $\underline{T} = \int dA \; \underline{r} \times \underline{\hat{n}}_1 (P_L - P_R)$

We calculate the torque in $\underline{y}$ direction, which is non-zero

$\hat{\underline{n}}_1 = \cos 60^\circ \hat{x} - \sin 60^\circ \hat{z} = \frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z}$.

$\underline{r} = (x, y, z) - (0, 0, z_2) = (x, y, z - z_2)$

~~the~~ relationship between x and z:

$\tan 60^\circ = \frac{x}{z - z_2} = \sqrt{3} \qquad x = \sqrt{3}(z - z_2)$

$\underline{r} \times \hat{\underline{n}}_1 = (\sqrt{3}(z - z_2)\hat{x} + y\hat{y} + (z - z_2)\hat{z}) \times (\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z})$

$= [-\frac{3}{2}(z_2 - z)\hat{y} + \frac{z - z_2}{2}\hat{y}] = 2(z - z_2)\hat{y}$

$P_L - P_R = \cancel{P_{atm}} - \rho_3 g \bar{z} - \rho_1 g (z - \bar{z}) - \cancel{P_{atm}} + \rho_4 g(\bar{z} - z_4) + \rho_1 g (z - \bar{z})$

$= (\rho_4 g - \rho_3 g)\bar{z} - \rho_4 g z_4 = g \, [(\rho_4 - \rho_3)\bar{z} - \rho_4 z_4]$

$\hat{y} \cdot \underline{T} = \int dA \; 2(z - z_2) \, g \, [(\rho_4 - \rho_3)\bar{z} - \rho_4 z_4]$

$= 2g \, [(\rho_4 - \rho_3)\bar{z} - \rho_4 z_4] \int (z - z_2) \, dA$

$dA = L \, dz / \sin 30^\circ = 2L \, dz$

$\hat{y} \cdot \underline{T} = 4gL \, [(\rho_4 - \rho_3)\bar{z} - \rho_4 z_4] \int_{z_2}^{\bar{z}} (z - z_2) \, dz$

$$\underline{T}_2 = 4gL\tau\,[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4]\ \frac{(z-z_2)^2}{2}\Big|_{z_2}^{\bar{z}}$$

$$= 4gL\tau\,[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4]\ \frac{(\bar{z}-z_2)^2}{2}\hat{y} = 2gL\tau[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4](\bar{z}-z_2)^2\hat{y}$$

3) $T_1 = T_2 + r_{12} \times F.$

$r_{12} = (0, 0, z_2 - z_1).$

$$\underline{F} = \int dA\ (P_L - P_R)\cdot \hat{n}$$

$$= \int 2L\,dz\ g[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4)].(\tfrac{1}{2}\hat{x} - \tfrac{\sqrt{3}}{2}\hat{z})]$$

$$\underline{F}\cdot\hat{x} = Lg\int_{z_2}^{\bar{z}} \tau[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4]\,dz$$

$$= Lg\,\tau[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4](\bar{z}-z_2)$$

$$\underline{r}_{12} \times (\underline{F}\cdot\hat{x}) = +\ (z_2-z_1)(\bar{z}-z_2)Lg\,\tau[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4]\hat{y}$$

~~Becau~~ The relationship between z_1 and z_2

$$(z_1-\bar{z})\tan 30^\circ = \frac{\bar{z}-z_2}{\tan 30^\circ} \Rightarrow 3(\bar{z}-z_2) = (z_1-\bar{z})$$

$$\hat{y}\cdot T_1 = (T_2 + r_{12}\times \underline{F})\cdot\hat{y}$$

$$= [(\rho_4-\rho_3)\bar{z} - \rho_4 z_4)gL\tau(\bar{z}-z_2)\ [2(\bar{z}-z_2) + z_2 - z_1]$$

$$= E(\rho_4-\rho_3)\bar{z} - \rho_4 z_4)gL(\bar{z}-z_2)\ [-(\bar{z}-z_2)]$$

$$= -\underline{T}_2$$

4) the torque around point I is zero

$$\frac{\rho_3-\rho_4}{6} z_1^3 + \frac{\rho_3-\rho_4}{3} \bar{z}^3 + \frac{\rho_4-\rho_3}{2} z_1 \bar{z}^2 + \frac{\rho_4}{2} z_4 z_1^2 + \frac{\rho_4}{2} z_4 \bar{z}^2 - \rho_4 z_1 z_4 \bar{z}$$

$$= 2gL\left[(\rho_4-\rho_3)\bar{z} - \rho_4 z_4 \right] (\bar{z}-z_2)^2$$