Math 104 - Midterm 1

Lecture 4, Fall 2011

September 30, 2011

Name: ______________________________

1. (10 points) Give an example of each of the following. You do not need to give any justification.

(a) A nonempty, bounded subset of $\mathbb{R}\backslash\mathbb{Q}$ with no infimum in $\mathbb{R}\backslash\mathbb{Q}$.

(b) A subspace of $\mathbb{R}$ containing $\mathbb{Z}$ in which $\{-4\}$ is open but $\{3\}$ is not.

(c) A nonempty subspace of $\mathbb{R}\backslash\mathbb{Q}$ which is complete in the metric space sense.

(d) An uncountable open subset of $(-1,1)\cap(\mathbb{R}\backslash\mathbb{Q})$ which is not all of $(-1,1)\cap(\mathbb{R}\backslash\mathbb{Q})$.

(e) A bounded sequence in $\{x \in \mathbb{R}\backslash\mathbb{Q} \mid -\pi < x < \pi\}$ which is not Cauchy in $\mathbb{R}$.

2. (15 points) For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ denote the function defined by

$$f_n(x) := \begin{cases} \frac{1}{3}x^n & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Compute the distance between $10f_n$ and f_n in $C_b([0,1])$ with respect to the sup metric and prove that your answer is correct. (Careful: note that $f_n(1) = 0$ and not 1 for all n.)

3. (15 points) Suppose that (x_n) is a sequence of real numbers and that $a, b \in \mathbb{R}$ with $a \neq 0$.

(a) If (x_n) converges to x and $ax_n + b \geq 0$ for all n, show that $(\sqrt{ax_n + b})$ converges to $\sqrt{ax + b}$.

(b) Give an example, with brief justification, where (x_n^4) converges but (x_n) does not.

(c) If (x_n^2) converges to 0, show that (x_n) converges to 0.

In (a) you must use only the definition of convergence and no other limit theorems.

4. (10 points) Suppose that (X, d) is a metric space and let $\{x_1, \ldots, x_n\}$ be a finite set of points of X. Show, using only the definition of open, that the set $X \backslash \{x_1, \ldots, x_n\}$ obtained by removing each x_i from X is open in X. (Draw a picture to get some intuition!)

5. (0 points) Draw a picture of your favorite closed subset of $\mathbb{R}^2$.