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Final Exam

May 9th, 2011

Prob. 1 a) $\text{Work} = \int_c^d p\,dV$

For adiabatic process: $pV^\gamma = \text{constant}$

$\therefore p_c V_c^\gamma = \text{constant} = C = p_d V_d^\gamma$

$p = C/V^\gamma$

$$W = \int_c^d \frac{C}{V^\gamma}\,dV$$

$$W = \frac{C}{1-\gamma}\cdot V^{1-\gamma}\Big|_{V_c}^{V_d} = \frac{C}{1-\gamma}\left(V_d^{1-\gamma} - V_c^{1-\gamma}\right)$$

$$W = \frac{p_c V_c^\gamma}{1-\gamma}\cdot\left[\left(V_c\cdot\left(\frac{p_c}{p_d}\right)^{1/\gamma}\right)^{1-\gamma} - V_c^{1-\gamma}\right] = \frac{p_c V_c}{1-\gamma}\left[\left(\frac{p_c}{p_d}\right)^{\frac{1-\gamma}{\gamma}} - 1\right]$$

$$\boxed{W = \frac{p_c V_c}{1-\gamma}\left[\left(\frac{p_c}{p_d}\right)^{\frac{1-\gamma}{\gamma}} - 1\right]}$$

b) efficiency: $\epsilon = \frac{W}{Q_H} = \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H}$

$\gamma = (d_f + 2)/d_f$

$d_f/2 = 1/(\gamma - 1)$

1st-law of thermo: $T\,dS = dQ = dE + p\,dV$

$\therefore \Delta Q = (\gamma-1)^{-1} N k_B \cdot \Delta T + p\cdot\Delta V$ for isobaric processes.

$$Q_H = \Delta Q_{cb} = (\gamma-1)^{-1} N k_B (T_c - T_b) + p_b (V_c - V_b)$$
$$= (\gamma-1)^{-1} p_b (V_c - V_b) + p_b (V_c - V_b) = \frac{\gamma\cdot p_b (V_c - V_b)}{\gamma - 1}$$

$$Q_L = -\Delta Q_{da} = \frac{\gamma\cdot p_a\cdot (V_d - V_a)}{\gamma - 1}$$ (same approach as above).

Adiabats: $p_a V_a^\gamma = p_b V_b^\gamma$

$V_a = V_b\cdot\left(\frac{p_b}{p_a}\right)^{1/\gamma}$

$p_c V_c^\gamma = p_d V_d^\gamma$

$V_c = V_d\left(\frac{p_d}{p_c}\right)^{1/\gamma} \leftrightarrow V_d = V_c\left(\frac{p_b}{p_a}\right)^{1/\gamma}$

$$\therefore \epsilon = 1 - \frac{\cancel{\gamma}\cdot p_a\cdot(V_d - V_a)}{\cancel{\gamma}\cdot p_b (V_c - V_b)} = 1 - \frac{p_a\cdot(p_b/p_a)^{1/\gamma}\cdot \cancel{(V_c - V_b)}}{p_b\cdot\cancel{(V_c - V_b)}} = \boxed{1 - \left(\frac{p_a}{p_b}\right)^{\frac{\gamma-1}{\gamma}} = \epsilon}$$

* Note: There should be a $(\gamma-1)$ in top + bottom that cancel.

2.

$\rho(r) = \rho_0 / r$

$$\int d^3r\, \rho(r) = \rho_0 (4\pi) \int_0^R \frac{r^2 dr}{r} = \rho_0 (4\pi) \int_0^R r\,dr = \frac{\rho_0 (4\pi) R^2}{2} = Q_0$$

$$\text{const} \Rightarrow Q = Q_0 = \boxed{\rho_0 = \frac{2Q}{4\pi R^2} = \frac{Q}{2\pi R^2}} \quad 3$$

Out

$$E_{out} = \frac{kQ}{r^2} = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}} \quad (\text{out}) \quad 3$$

$$U_{out} = \int \frac{1}{2}\varepsilon_0 |E_{out}|^2 d^3r = \frac{1}{2}\varepsilon_0 \frac{1}{(4\pi\varepsilon_0)^2} Q^2 (4\pi) \int_R^\infty dr \frac{r^2}{r^4}$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)_R^\infty = \boxed{\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{R}} \quad 3$$

In,

$$\oint \vec{E}\cdot d\vec{a} = 4\pi R^2 E = \frac{Q_{in}}{\varepsilon_0}$$

$$E = \frac{Q_{in}}{4\pi\varepsilon_0 r^2} \quad ; \quad Q_{in} = 4\pi\int_0^r r^2 \rho(r)\,dr = 4\pi\rho_0 \int_0^r r\,dr = 4\pi\left(\frac{Q}{2\pi R^2}\right)\int_0^r r\,dr$$

$$= \frac{2Q}{R^2}\frac{r^2}{2} = \frac{Q r^2}{R^2} \quad (4)$$

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 R^2}} \quad (\text{in}) \quad 3$$

$$U_{in} = \frac{1}{2}\varepsilon_0 \int |E|^2 d^3r = \frac{1}{2}\varepsilon_0 \frac{Q^2}{(4\pi\varepsilon_0)^2 R^4}\frac{4}{3}\pi R^3 = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{3R} = \boxed{\frac{1}{2}\frac{Q^2}{12\pi\varepsilon_0 R}} \quad 3$$

Total

$$U = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\underbrace{\left[\frac{1}{R} + \frac{1}{3R}\right]}_{\frac{4}{3R}} = \boxed{\frac{Q^2}{6\pi\varepsilon_0 R}}$$

Errors

(+1) (only) for $U = QV$

(+2) (only) for $U = \sum U_{ij}$, w/ $U_{ij} = \frac{k q_i q_j}{r_{ij}}$

(+2) (only) if relates $U = \frac{1}{2}\varepsilon_0 |E|^2$, but nothing else

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③

a) inside   Just: (+2)   Ans. (+2)   (wrong: 0)

b) $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$ Just: (+1)   Ans: (+2)   (−1 for vector)

c) $\frac{P}{V} = \frac{Q \times d}{V} = \frac{\sigma \times A \times a}{A \times a} = \sigma$ (3)   (−1 for vector)

pointing up!

(=1) if wrong but makes some sense

(=2) if Vol = A·t instead of A·a

(=0) if wrong formula, no idea...

Problem 4

(a) Charges are at rest, find electric force.

$$Gauss'\,Law: \int \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \quad \rightarrow \quad E_{cyl}(2\pi r l) = \frac{\lambda l}{\epsilon_0}$$

$$E = \frac{\lambda}{2\epsilon_0 \pi r} \quad \rightarrow \quad \vec{F} = q\vec{E} = \frac{q\lambda}{2\epsilon_0 \pi r}\hat{r}$$

(b) Moving charge creates magnetic field. Use Lorentz force law for net force

$$Ampere's\,Law: \int \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \quad \rightarrow \quad B(2\pi r) = \mu_0 I \quad \rightarrow \quad \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

$$I = \frac{dq}{dt} = \frac{dq}{dx}\frac{dx}{dt} = \lambda v \quad \rightarrow \quad \vec{B} = \frac{\mu_0 \lambda v}{2\pi r}\hat{\theta}$$

$$Lorentz\ Force\ Law: \vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = q(E\hat{r} + (v\hat{z})\times(B\hat{\theta})) = q(E - vB)\hat{r}$$

$$\vec{F} = q\left(\frac{\lambda}{2\epsilon_0 \pi r} - \frac{\mu_0 \lambda v^2}{2\pi r}\right)\hat{r}$$

$$Eliminate\,\mu_0: \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \rightarrow \quad \mu_0 = \frac{1}{\epsilon_0 c^2}$$

$$\vec{F} = \frac{q\lambda}{2\epsilon_0 \pi r}\left(1 - \frac{v^2}{c^2}\right)\hat{r}$$

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Problem [illegible] Solution

a. The back EMF of an inductor [illegible] $\mathcal{E} = -L\frac{dI}{dt}$, so in steady state the solenoid will act like a wire with internal resistance. (1 point)

$$R = \rho \frac{l}{A}$$

$$l = 2\pi r_s N = \frac{2\pi r_s l_s}{w}$$ ([illegible] point)

$$A = wh$$ ([illegible] point)

[illegible] $$[illegible] = \frac{\mathcal{E}_0 w^2 h}{\rho l_s 2\pi r_s}$$ (1 point)

b. $$L = \frac{\Phi}{I}$$ ([illegible] point)

The magnetic field [illegible] can be found using ampere's law, and approximating the solenoid to be a [illegible], i.e. uniform surface current flowing cylindrically, and $l_s \gg r_s$.

$$B = \mu_0 \frac{N}{l_s} I$$ (2 points)

$$\Rightarrow \Phi = N A_s B = \frac{N^2 A_s \mu_0 I}{l_s}$$ (2 points)

$$\Rightarrow L = \frac{N^2 A_s \mu_0}{l_s} = \frac{A_s \mu_0 l_s}{w^2}$$

c. The time constant can be derived by treating this as an LR circuit, for [illegible] Kirchoff's laws [illegible]

$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0$$

$$\Rightarrow \frac{\mathcal{E}_0}{R} - I - \frac{L}{R}\frac{dI}{dt} = 0$$

$$\Rightarrow \tau = \frac{L}{R}$$

It was not [illegible] (3 points)

[illegible] plugging in [illegible] L and [illegible] b and a

$$\tau = [illegible]$$ (3 points)

$$[illegible] = \frac{\mu_0 [illegible]}{2\rho}$$ (2 points)

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Problem 6 Solution

$$Q = \int dQ = \int \frac{dQ}{dt}\,dt = \int I\,dt \qquad \text{(3 points)}$$

$$I = \frac{\mathcal{E}}{R} = \frac{1}{R}\frac{d\Phi}{dt} \Longrightarrow Q = \int \frac{1}{R}\frac{d\Phi}{dt}\,dt \qquad \text{(2 points)}$$

$$= \frac{1}{R}\int d\Phi \qquad \text{(2 points)}$$

$$= \frac{1}{R}\Delta\Phi$$

to maximize the charge that flows, clearly the change in flux must be largest, i.e. rotating from parallel to perpendicular with the magnetic field.

$$\Rightarrow Q = \frac{1}{R}\left(\pi r^2 B - 0\right) \qquad \text{(2 points)}$$

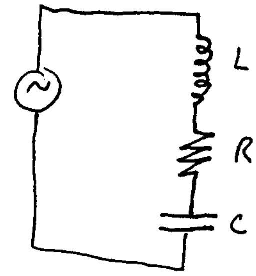
the number of electrons then is

$$N_e = \frac{Q}{e} = \frac{\pi r^2 B}{Re} \qquad \text{(1 point)}$$

where e is the charge of one electron.

Note that it is not necessary to assume that the process happens at constant angular velocity, nor is it necessary to approximate $\frac{d\Phi}{dt} \approx \frac{\Delta\Phi}{\Delta t}$.

Problem 7



$V_{L,max} = I_0 X_L$

$V_{R,max} = I_0 R$

$V_{C,max} = I_0 X_C$

$X_L = \omega L$ (+1), $X_C = \frac{1}{\omega C}$ (+1)

[10 pts] a) $V = V_0 \cos(\omega t + \phi)$

First, solve for I_0 (amplitude of current)

amplitude of voltage of power source

$\hookrightarrow V_0 = I_0 \cdot Z$ where $Z = \sqrt{R^2 + (X_L - X_C)^2}$ (+1)

Z ↑ impedance

$$\Rightarrow I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \quad (+3)$$

amplitude of voltage across capacitor is:

$$V_{C,max} = I_0 X_C \overset{(+2)}{=} \left(\frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}\right) X_C = \left(\frac{X_C}{\sqrt{R^2 + (X_L - X_C)^2}}\right) V_0$$

$$\Rightarrow \boxed{V_{C,max} = \frac{V_0}{\omega C \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}} \quad (+2)$$

[5 pts] b) at resonance, $\omega = \frac{1}{\sqrt{LC}}$ (+1) $\Rightarrow X_L = X_C = \sqrt{\frac{L}{C}}$ (+1)

so $V_{C,max} = \frac{X_C}{R} V_0$ (+1) $= \frac{1}{R}\sqrt{\frac{L}{C}}\, V_0$

$$\Rightarrow \boxed{\frac{V_{C,max}}{V_0} = \left(\frac{L}{R^2 C}\right)^{1/2} > 1} \quad \text{since } R^2 C < L$$

(+2)