Mathematics 54.1
Final Exam, 12 May 2011
180 minutes, 90 points

NAME: ______________________ **ID:** ______________________

GSI: ____

INSTRUCTIONS:
You must justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.

There is an Extra Credit question. It will not be added to the score, but a substantially complete answer will bump your exam grade by one step.

This is a CLOSED BOOK examination, NO NOTES and NO CALCULATORS are allowed.
NO CELL PHONE or EARPHONE use is permitted.
Please turn in your finished examination to your GSI before leaving the room.

Q1	
Q2	
Q3	
Q4	
Tot	
Letr	
Xtra	

Question 1. (40 points) Choose the correct answer, worth 2 points each. You need not justify your answer. There is no penalty for wrong answers, but no credit is given if more than one answer is circled.
Please transfer your choices into the table at the end of the question.

1. The *left nullspace* of an $m \times n$ matrix A is

   (a) The set of vectors $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \mathbf{0}$
   (b) The set of vectors $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x}A = \mathbf{0}$
   (c) The set of vectors $\mathbf{x} \in \mathbb{R}^m$ with $A\mathbf{x} = \lambda\mathbf{x}$
   (d) The set of vectors $\mathbf{x} \in \mathbb{R}^m$ with $A\mathbf{y} = \mathbf{x}$

2. The following vector is in the left nullspace of $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$:

   (a) $[1, 2, 1]$
   (b) $[-1, -2, 1]$
   (c) $[1, 1, 2]$
   (d) $[1, -2, 1]$

3. A *linear transformation* is defined as a map $T : V \to W$ such that

   (a) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$
   (b) $T(c\mathbf{x} + d\mathbf{y}) = T(c\mathbf{x}) + T(d\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c, d \in \mathbb{R}$
   (c) $T(c\mathbf{x} + d\mathbf{y}) = T(c\mathbf{x}) + T(d\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c, d \in \mathbb{R}$, and $T(\mathbf{0}) = \mathbf{0}$
   (d) $T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c, d \in \mathbb{R}$

4. The following map $T : \mathbb{R}^2 \to \mathbb{R}$ is NOT a linear transformation:

   (a) $T([x, y]^T) = x + y$
   (b) $T([x, y]^T) = x + |y|$
   (c) $T([x, y]^T) = x - y$
   (d) $T([x, y]^T) = x - x^2 + y^2 + (x + y)(x - y)$

5. A number λ is an *eigenvalue* of an $n \times n$ matrix A if and onlly if:

   (a) $\det(A - \lambda I_n) = 0$
   (b) λ is a pivot of A
   (c) $A - \lambda I_n$ is non-singular
   (d) $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$

6. The following is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$:

   (a) 1
   (b) 2
   (c) 3
   (d) 5

7. A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is linearly independent if and only if:

   (a) It forms a basis of $\mathbb{R}^n$
   (b) Any two vectors in it are linearly independent
   (c) $\mathbf{v}_1 + \cdots + \mathbf{v}_k = \mathbf{0}$ implies that each $\mathbf{v}_i = \mathbf{0}$
   (d) $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ implies that each $c_i = 0$

8. The following collection is linearly independent:

   (a) $[1, 2, 3], [2, 4, 5], [3, 6, 7]$
   (b) $[1, 1, 1], [1, 2, 3], [3, 4, 5]$
   (c) $[1, 2, 3], [2, 4, 6]$
   (d) $[1, 1, 2], [2, 3, 3], [3, 4, 5]$

9. The *rank* of a matrix is

(a) The number of rows

(b) The dimension of its column space

(c) The number of columns

(d) The dimension of its left column space

10. The following matrix has rank 2:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}$

11. A subspace S of $\mathbb{R}^n$ is the *orthogonal complement* of a subspace T if and only if

(a) Some vector in S is orthogonal to some vector in T

(b) All vectors in S are orthogonal to each other, as are all vectors in T

(c) Every vector in S is orthogonal to every vector in T

(d) S contains precisely the vectors orthogonal to every vector in T

12. The following subspace in $\mathbb{R}^4$ is the orthogonal complement of

$$T := \{[x_1, x_2, x_3, x_4]^T \mid x_1 - x_3 = 0, x_1 + x_2 - x_3 - x_4 = 0\} :$$

(a) $\text{Span}([1, 2, -1, -2]^T, [2, 3, -2, -2]^T)$

(b) $\text{Span}([1, 2, -1, -2]^T, [2, 3, -2, -3]^T, [3, 4, -3, -4]^T)$

(c) $\text{Span}([1, 2, -1, -2]^T, [-1, -2, 1, 2]^T)$

(d) $\text{Span}([1, 2, -1, -2]^T)$

13. A *least-squares solution* of the system $A\mathbf{x} = \mathbf{b}$ is

(a) A vector $\mathbf{x}$ such that $A\mathbf{x}$ has minimum length

(b) A vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$

(c) A vector $\mathbf{x}$ such that $A\mathbf{x} - \mathbf{b}$ has minimum length

(d) A vector $\mathbf{x}$ such that $A(\mathbf{x} - \mathbf{b})$ has minimum length

14. The least-squares solution to $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is

(a) $\mathbf{x} = 1$

(b) $\mathbf{x} = 2$

(c) $\mathbf{x} = 3$

(d) $\mathbf{x} = 4$

15. A matrix is *orthogonal* if

(a) It is square and its columns are mutually orthogonal

(b) It is square and its columns are mutually orthogonal and have length 1

(c) It is square, invertible and agrees with its inverse

(d) Its columns are mutually orthogonal and have length 1, whether or not it is square

16. The following matrix is NOT orthogonal:

(a) $[1]$

(b) $[2]$

(c) $\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

(d) $\begin{bmatrix} .6 & .8 \\ .8 & .-6 \end{bmatrix}$

17. A matrix A is *diagonalizable* if

(a) We can find matrices S and D with $AS = SD$ and D diagonal

(b) We can find matrices S and D with $AS = SD$, D diagonal and S invertible

(c) We can find matrices S and D with $AS = SD$, D diagonal and A invertible

(d) A has eigenvectors

18. The following matrix is NOT diagonalizable:

(a) $\begin{bmatrix} .2 & 1 \\ 0 & .1 \end{bmatrix}$

(b) $\begin{bmatrix} .2 & 1 \\ 0 & .2 \end{bmatrix}$

(c) $\begin{bmatrix} .2 & 1 \\ 0 & .3 \end{bmatrix}$

(d) $\begin{bmatrix} .2 & 1 \\ .1 & .2 \end{bmatrix}$

19. The *characteristic polynomial* of an 2×2 matrix A is

(a) $\det(\lambda I_2 + A)$

(b) $\lambda^2 - \lambda \mathrm{Tr}(A) + \det(A)$

(c) $\det(\lambda I_2) - \det(A)$

(d) $\lambda^2 + \lambda \mathrm{Tr}(A) + \det(A)$

20. The *exponential* of a square matrix A is

(a) The sum of the series $I + A + A^2 + A^3 + \dots$

(b) The sum of the series $I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$

(c) The matrix whose (i, j)-entry is $\exp(A_{ij})$, where A_{ij} is the (i, j)-entry of A

(d) The diagonal matrix with entries e^{λ_i}, where the λ_i are the eigenvalues of A

Q1	a	b	c	d
Q2	a	b	c	d
Q3	a	b	c	d
Q4	a	b	c	d
Q5	a	b	c	d
Q6	a	b	c	d
Q7	a	b	c	d
Q8	a	b	c	d

Q9	a	b	c	d
Q10	a	b	c	d
Q11	a	b	c	d
Q12	a	b	c	d
Q13	a	b	c	d
Q14	a	b	c	d
Q15	a	b	c	d
Q16	a	b	c	d

Q17	a	b	c	d
Q18	a	b	c	d
Q19	a	b	c	d
Q20	a	b	c	d

Question 2. (16 points)
(a) Describe Lagrange's method of "Variation of parameters" for solving the second-order inhomogeneous ODE

$$x''(t) + a_1(t)x'(t) + a_0(t)x(t) = g(t).$$

(You need not prove that the method is correct, just describe it clearly.)

(b) By Lagrange's method, or other tools that you know, solve the ODE

$$\frac{1}{2}x'' + 2x = \tan(2t) \qquad (\text{where } -\frac{\pi}{4} < t < \frac{\pi}{4})$$

Hint: you might find the formula $\displaystyle\int \frac{dt}{\cos 2t} = \frac{1}{2}\log\frac{\cos t + \sin t}{\cos t - \sin t}$ *useful.*

Question 3. (18 points)
(a) Find the Fourier cosine series for the function $f(x) = \sin x$ on the interval $[0, \pi]$.
You might find the addition formula $\sin(a + b) = \sin a \cos b + \cos a \sin b$ *useful.*

(b) Specialize your Fourier series to $x = \pi/2$ to get an interesting identity.
Recall that $\sin \frac{\pi}{2} = 1$ *and* $\cos n\pi = (-1)^n$.

Question 4. (16 points)

(a) Write the general form of d'Alembert's solution to the wave equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ for a function $u(x,t)$, with $x, t \in \mathbb{R}$. Briefly explain how the ingredients of the solution can be found from the initial conditions.

(b) Solve the equation explicitly, subject to the initial conditions $u(x,0) = \sin^2 x$ and $\partial u/\partial t(x,0) = \cos x$.
Remark: The initial conditions are 2π-periodic. This opens another approach to the question, if for some reason you are having trouble with d'Alembert's method.

Extra Credit Question

Only answers that are substantially complete will be considered; there is no partial credit for deficient answers. Therefore, you are advised to work on this question only after completing the others.

(a) Write down the general solution of the vector-valued ODE

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1.4 & 1.6 \\ -.8 & -.2 \end{bmatrix} \cdot \mathbf{x}$$

(b) Draw a 'phase diagram' of this ODE, roughly sketching a few trajectories.

(c) Which axis is the first to be crossed, if we start with the initial value $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

THIS PAGE IS FOR ROUGH WORK (not graded)