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8:10-9:00, Friday, February 23, 2007

ME 106 FLUID MECHANICS

EXAM 1 – open book, open notes, no external communication

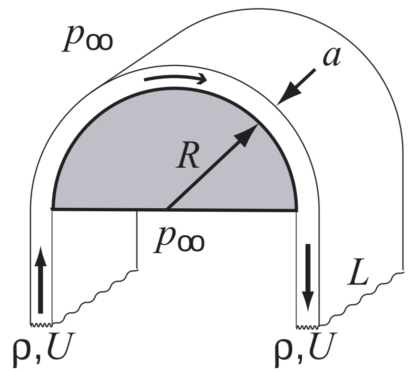
1.((10+10)+15=35%)

You have done the egg experiment. Let us now consider a simpler version of it. Suppose a thin sheet of water is skimming a semi-cylindrical shell of radius R. The sheet has a thickness of $a \ll R$, a width of L and wraps around the shell. Assume uniform density ρ and flow speed U throughout the sheet. Ignore gravity and viscosity. Note that pressure is atmospheric on all open surfaces in contact with air.

Determine the force exerted on the shell by the water sheet using two independent methods:

(a) First, determine the pressure distribution and then integrate it over the curved surface.

(b) Second, use the momentum integral.



SOLUTION

(a)

$$\frac{dp}{dn} \approx \frac{\Delta p}{a} = \rho \frac{U^2}{R} \Longrightarrow \Delta P = \rho a U^2/R, \quad \text{uniform pressure}$$

$$F = \Delta P \times 2R \times L = 2\rho U^2 A \quad \text{where} \quad A = aL$$

(b) Force on fluid

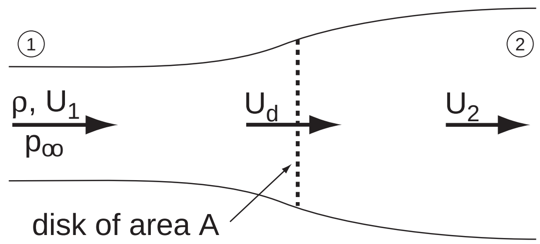
$$\mathbf{F}_f = \int_S \rho \mathbf{u}(\mathbf{u} \cdot d\mathbf{A}) = \rho(0, U)\,[(0, U) \cdot (0, -A)] + \rho(0, -U)\,[(0, -U) \cdot (0, -A)] = (0,\ -2\rho U^2 A)$$

Force on shell

$$\mathbf{F}_s = -\mathbf{F}_f = (0,\ +2\rho U^2 A)$$

2.(5+10+10+5+5=35%)

A windmill may be modeled as a disk of area A which extracts energy from the incoming wind stream of velocity U_1. As indicated in the figure, air is slowed down by the disk from U_1 at far upstream to U_2 at far downstream. The air velocity through the disk is U_d. Assume constant air density of ρ and ignore viscous effects and all losses. Also assume that flow is purely axial.



(a) Write the mass flow rate $\dot{m}$ through the stream tube far upstream ①, at the disk, and far downstream ②.

(b) Using the momentum balance far from the disk, determine the axial force on the disk.

(c) Writing Bernoulli's equation between far upstream and upstream side of the disk; and between downstream side of the disk and far downstream, deduce the axial force again.

(d) By combining your results from (b) and (c), determine U_d. Then, determine the kinetic energy extraction rate in terms of U_1 and U_2.

(e) Finally, determine the maximum power extraction.

SOLUTION

Mass flow rate through the stream tube

$$\dot{m} = \rho A U_d \tag{1}$$

From momentum balance at far distances from the disk where the ambient pressure is p_∞, the drag on the disk is

$$D = \dot{m}(U_1 - U_2) = \rho A U_d (U_1 - U_2) \tag{2}$$

Writing Bernoulli's equation between far upstream and upstream side of the disk; and between downstream side of the disk and far downstream, we obtain for drag

$$D = \Delta p A = \frac{1}{2}\rho A (U_1^2 - U_2^2) \tag{3}$$

Equating Eq. 2 and Eq. 3, we determine U_d as

$$U_d = \frac{1}{2}(U_1 + U_2) \tag{4}$$

We now write the rate of total kinetic energy extracted at the disk

$$\Delta \dot{KE} = P = \frac{1}{2}\dot{m}(U_1^2 - U_2^2) = \frac{1}{2}\rho A U_d (U_1^2 - U_2^2) = \frac{1}{4}\rho A (U_1 + U_2)(U_1^2 - U_2^2) \tag{5}$$

The maximum extraction occurs when

$$\frac{dP}{dU_2} = 0 \implies U_2 = \frac{1}{3}U_1 \implies P_{max} = \frac{8}{27}\rho A U_1^3 \tag{6}$$

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3. (5+10+10+5=30%)

Consider the following velocity field in cylindrical polar coordinates

$$\mathbf{u} = (u_r, u_\theta) = \left(\frac{cos(2\theta/3)}{r^{1/3}}, -\frac{sin(2\theta/3)}{r^{1/3}}\right).$$

(a) By simply examining the velocity field above, sketch the streamlines passing through $(r, \theta) = (1, 0)$ and $(r, \theta) = (1, 3\pi/2)$

(b) Find the general form of the equation for the streamlines by integrating $d\mathbf{x} \times \mathbf{u} = \mathbf{0}$

(c) Sketch the streamline passing through $(r, \theta) = (1, 3\pi/4)$

(d) Sketch a few streamlines, including those from (a) to deduce the overall flow pattern.

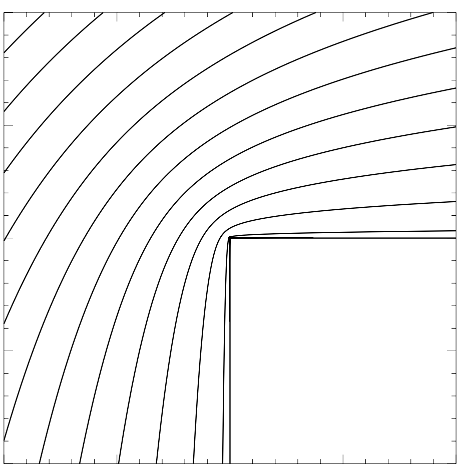
Hints:

1. In cylindrical coordinates the vector equation for streamlines, $d\mathbf{x} \times \mathbf{u} = \mathbf{0}$, reads

$$\frac{dr}{u_r} = \frac{r d\theta}{u_\theta}$$

2. $$\int \frac{\cos x}{\sin x} dx = ln(\sin x)$$

SOLUTION



$$\frac{dr}{u_r} = \frac{r d\theta}{u_\theta} \tag{7}$$

$$\frac{dr}{r^{-1/3} cos(2\theta/3)} = \frac{r d\theta}{-r^{-1/3} sin(2\theta/3)} \tag{8}$$

$$\int \frac{dr}{r} = -\int \frac{d\theta}{tan(2\theta/3)} \tag{9}$$

$$ln(r) = -(3/2) ln(sin(2\theta/3)) + constant \tag{10}$$

$$(2/3) ln(r) + ln(sin(2\theta/3)) = ln(r^{2/3} sin(2\theta/3)) = constant \tag{11}$$

$$r^{2/3} sin(2\theta/3) = constant \tag{12}$$