momentum conservation:

\[
\begin{align*}
L & \quad mV_A = m u_A + m u_B \\
E & \quad \frac{P = E_f - E_i}{E_i} = \frac{\frac{1}{2} m u_A^2 + \frac{1}{2} m u_B^2 - \frac{1}{2} m V_A^2}{\frac{1}{2} m V_A^2}
\end{align*}
\]

\[
\Rightarrow \quad \frac{u_A}{\sqrt{A}} + \frac{u_B}{\sqrt{A}} = 1
\]

\[
\ln\left(\frac{u_A}{\sqrt{A}}\right) + \ln\left(\frac{u_B}{\sqrt{A}}\right) - 1 = P
\]

\[
\Rightarrow \quad \frac{u_A}{\sqrt{A}} + \frac{1 - \frac{u_A}{\sqrt{A}}}{\sqrt{A}} - 1 = P
\]

\[
2\left(\frac{u_A}{\sqrt{A}}\right)^2 - 2 \left(\frac{u_A}{\sqrt{A}}\right) - P = 0
\]

\[
\frac{u_A}{\sqrt{A}} = \frac{2 \sqrt{4 + 8P}}{4} = \frac{1 - \sqrt{1 + 2P}}{2}
\]

\[
\frac{u_B}{\sqrt{A}} = \frac{1 + \sqrt{1 + 2P}}{2}
\]
Energy Conserved

\[ E_1 = E_2 \]

Rolling w/o slipping

\[ V_p = V_s - \omega R = 0 \Rightarrow \omega = V_s/R \]

\[ M_s g h = \frac{1}{2} M_s V_s^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} M V_m^2 \]

Play 2, 3 into 1

\[ M_s g h = \frac{1}{2} M_s V_s^2 + \frac{1}{2} I \frac{V_s^2}{R^2} + \frac{1}{2} M \frac{M_s^2}{M} V_s^2 \Rightarrow M_s g h = \frac{1}{2} M_s V_s^2 \left( 1 + \frac{I}{\frac{M_s^2}{M}} + \frac{M_s}{M} \right) \]

\[ V_s = \sqrt{\frac{2g h}{1 + \frac{I}{\frac{M_s^2}{M}} + \frac{M_s}{M}}} \]

Check: \( I = 0, \ M \rightarrow \infty \ \Rightarrow \ V_s = \sqrt{2g h} \)

Play into 4

\[ V_m = \frac{M_s}{M} \sqrt{\frac{2g h}{\frac{I}{\frac{M_s^2}{M}} + \frac{M_s}{M}}} \]
\(* 3. \)

a) \( dU = (d\lambda \cdot g) y = \lambda \, dy \, g_y \)

\[ \text{infinitesimal potential energy of the cord} \]

\[ \text{btw} \ y \ \text{and} \ y+dy \], \text{and} \ \lambda = \frac{M}{L} \]

\[ U(Y) = \int dU = \frac{M}{L} \int_y^{L-Y} y \, dy \]

\[ = Mg \cdot \frac{1}{L} \cdot \frac{(L-Y)^2}{2} = \frac{Mg}{2L} (L-Y)^2 \]

or using center of mass for (L-Y) cord.

b) We use energy conservation.

\[ E_i = U(Y=0) + \frac{1}{2} M [V(Y=0)] = U(Y=0) \]

as \ V(Y=0) = 0.

\[ E_i = E(Y) = U(Y) + \frac{1}{2} M V(Y)^2 \]

then

\[ V(Y) = \sqrt{\frac{2}{M} (U(Y=0) - U(Y))} \]

\[ = \sqrt{\frac{2}{M} \left( \frac{Mg}{2L} \frac{L^2}{2} - \frac{Mg}{2L} (L-Y)^2 \right)} = \sqrt{\frac{g}{L} \left( \frac{L^2}{2} - (L-Y)^2 \right)} \]
Midterm 2 Solution: Problem 4
Physics 7A, UC Berkeley, Spring 2011, Prof. A. Speliotopoulos
Grader: Aaron Alpert

Linear Dynamics. We draw the free body diagram as shown, making careful note of the fact that we have defined clockwise as positive rotation. We write down the linear dynamics equation,
\[ \sum F_x = F - f_s = ma. \] (1)

Rotational Dynamics. The magnitude of the torque about the CM due to \( F \) is \( rF \sin \phi \), as usual, but we note that in this definition, \( \phi \) is the angle between vectors \( \ell \) and \( F \). As shown in the diagram, \( \theta \) is the angle between \( \ell \) and the vertical. Thus, we must use \( \ell \cos \theta \), which is the perpendicular length between the disk’s center and \( F \). For the frictional force, the corresponding torque’s magnitude is \( f_s R \). Both forces tend to cause clockwise rotation, so,
\[ \sum \tau = F \ell \cos \theta + f_s R = I \alpha. \] (2)
[If you defined counter-clockwise as positive, the left hand side would have a negative.]
For a disk, \( I = \frac{1}{2} MR^2 \), so
\[ \sum \tau = F \ell \cos \theta + f_s R = \frac{1}{2} MR^2 \alpha. \] (3)

Kinematic Relations. The disk rolls without slipping. We note that if counter-clockwise were defined as positive, the following equation would have a negative sign on one side.
\[ a = R \alpha \] (4)

Solution. We solve equations (1) (3) and (4) for the unknowns \( f_s, \alpha, \) and \( a \). Begin by using (4) to eliminate \( a \) from (1)
\[ F - f_s = ma = m(R\alpha) \quad \rightarrow \quad \alpha = \frac{F - f_s}{mR} \] (5)
Substitute the expression for \( \alpha \) in (5) into (3)
\[ F \ell \cos \theta + f_s R = \frac{1}{2} mR^2 \left( \frac{F - f_s}{mR} \right) = \frac{1}{2} R(F - f_s) \] (6)
\[ \frac{3}{2} f_s R = F \left( \frac{1}{2} R - \ell \cos \theta \right) \quad \rightarrow \quad f_s = \frac{2}{3} F \left( \frac{1}{2} - \frac{\ell}{R} \cos \theta \right) \] (7)
Solution

FBD

\[ \begin{align*}
  T & \quad m_A \\
  m_A \cdot g & \quad m_B \\
  m_B \cdot g &
\end{align*} \]

Net force + net torque equations

1) \( m_A \cdot T - m_A g = m_A \alpha_A \)

2) \( T - m_B g = m_B \alpha_B \)

3) \(-T \cdot R = (\frac{1}{2} m_B R^2) \cdot \alpha_B\) \hspace{1cm} \text{net torque clockwise is negative}

Constraint for rolling without slipping

4) \( \alpha_B - \alpha_B \cdot R = -\alpha_A \) \hspace{1cm} \text{absolute acceleration of point on disk in contact with rope is accelerating at } -\alpha_A

4 \text{ unknowns } T, \alpha_B, \alpha_A, \alpha_B

4 \text{ equations! solve for } \alpha_B \text{ and } \alpha_A
1) \[ 2T - (m_A + m_B)g = m_A a_A + m_B a_B \]

2) \[ (m_B - m_A)g = m_A a_A - m_B a_B \]

4) \[ T \cdot R = \left( \frac{1}{2} m_B R^2 \right) \frac{(a_B + a_A)}{R} \]

5) \[ T = \frac{1}{2} m_B (a_B + a_A) \]

-2 \cdot 5) \rightarrow 1)+2)\]

\[ - (m_A + m_B)g = (m_A + m_B) a_A + 2m_B a_B \]

\[ (m_B - m_A)g = m_A a_A - m_B a_B \]

2 knowns, \( a_A \) & \( a_B \), 2 equations!

\[ a_A = \frac{m_B - 3m_A}{3m_A + m_B} \cdot g \]

\[ a_B = -\left( \frac{m_A + m_B}{3m_A + m_B} \right) \cdot g \]