1. Let matrix B be defined by

$$B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix},$$

$\frac{8}{2} - \frac{9}{2}$

$4 - 3 \cdot \frac{3}{2}$

and let **B** be a basis consisting of columns of B. Let $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be a vector in $\mathbf{R}^2$. Find the **B**-coordinates of x.

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[X]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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2. Let matrix A be defined by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

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(a) Find all the eigenvalues of A.

(b) Diagonalize A if possible; otherwise show why A is not diagonalizable.

(a) $\begin{bmatrix} 2-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & -\lambda \end{bmatrix}$

$(2-\lambda)(1-\lambda)(-\lambda) - 2(2-\lambda) = 0$

$(2-\lambda)[-\lambda + \lambda^2 - 2] = 0$

$(2-\lambda)(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

(b) $\lambda = 2$ $\begin{bmatrix} 2-2 & -1 & 0 \\ 0 & 1-2 & 1 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{cc} A & \vec{0} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

$Ax = 0$

e-vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x$

$\lambda = -1$ $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

A is not diagonalizable, because $\begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

A is $n \times n$, $n = 3$ and there are only 2 eigenvectors.

For a 3×3 matrix to be diagonalizable, It must have 3 lin. ind. eigenvectors, which A does not.

$\to$ A is not diagonalizable.

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3. Let A be an $n \times n$ matrix.

11 (a) Let u be an eigenvector of A corresponding to an eigenvalue λ, and let H be the line in $\mathbf{R}^n$ through u and the origin. Explain why H is invariant under A in the sense that Ax is in H whenever x is in H.

10 (b) Let K be a one-dimensional subspace of $\mathbf{R}^n$ that is invariant under A. Explain why K contains an eigenvector of A.

(a) If $x \in H$, then $x = cu$, c is a constant. So x is some eigen vector for A. So why? $Ax = \lambda x = \lambda cu$. λ is a constant, and any constant multiple of u is an eigen vector of A, and is also $\in$ of H. Thus, $Ax \in H$. if $x \in H$

$Ax = \lambda x \in H$

$Ax \in H$.

(b) If k is invarient under A then there exists an x → a non zero x such that $Ax \in K$ when $x \in k$. This means that Ax is some constant multiple λ of x, since k is one dimensional, and spanned by x. So $Ax = \lambda x$. This is the definition of an eigen vector, for A as $x \in k$ and $Ax = \lambda x \in k$. k must then contain an eigenvector

$x \in H$, $Ax \in H$

$Ax = \lambda x$ since H is of dim 1, and $\{x\}$ is a basis for H.

4. (a) Let subspace $\mathbf{W} = \mathbf{span}(u, v)$, where

$$u = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 10 \\ 1 \\ 3 \end{pmatrix}.$$

Find an orthonormal basis for $\mathbf{W}$ using Gram-Schmidt process.

(b) Let $A \in \mathbf{R}^{m \times n}$ be an $m \times n$ matrix and $b \in \mathbf{R}^m$ be an m-dimensional vector. Show that the normal equation

$$A^T A x = A^T b$$

has a solution for any such A and b.

$64 + 9 + 1$

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(a) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $u_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$V_2 = \begin{bmatrix} 10 \\ 1 \\ 3 \end{bmatrix} - \frac{10+2+6}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 8/\sqrt{74} \\ -3/\sqrt{74} \\ -1/\sqrt{74} \end{bmatrix}$

Orthonormal basis: $\left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} 8/\sqrt{74} \\ -3/\sqrt{74} \\ -1/\sqrt{74} \end{bmatrix} \right\}$ 10

(b) $A^TAx = A^Tb$ is true if A^TA and A^T share a column space, Then

$A^TAx = C$
$A^Tb = C$ $\rightarrow$ $A^TAx = A^Tb$ for some x and b.

Show $A^TA \subseteq A^T$

Let $A^TAx = b$

$Ax = y \in R^n$

$A^Ty = b$

So $A^TA \subseteq A^T$

Show $A^T \subseteq A^TA$

Let $A^Tx = b$

By vector decomposition theorem:

$x = A\hat{x} + \hat{p}, \quad \hat{p} \perp A,$

$b = A^T(A\hat{x} + \hat{p}) = A^TA\hat{x} + A^T\hat{p}$

$= A^TAx + 0$

$A^TAx = b$

$A^T \subseteq A^TA$

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Thus $A^TAx = A^Tb$ for some x, b

And $A^TAx = A^Tb$ has a solution for any such A and b.