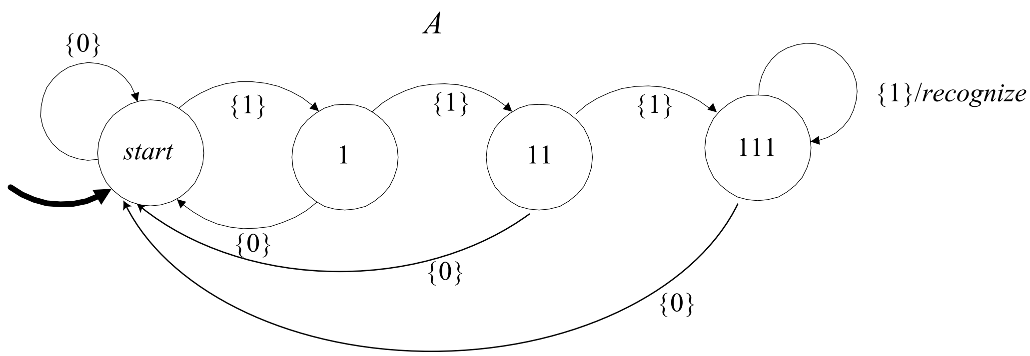
EECS 20. Final Exam Solution
May 20, 2003.

1. **15 points. 5 points for (a), 10 points for (b)**

   (a) The deterministic machine A is like the *CodeRecognizer* machine studied in the text and in the homework.



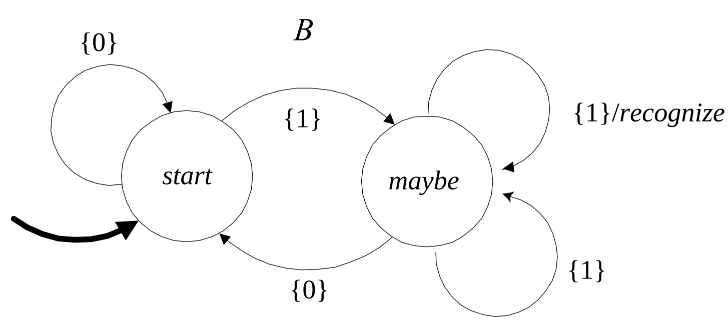
   Let x denote an input signal and y the corresponding output signal. Complete the expression for $y(n)$ below, ignoring stuttering inputs, (i.e. replace the $\cdots$ by an expression involving x)

$$\forall x \in \textit{InputSignals}, \qquad \forall n \in \textit{Naturals}_0,$$

$$y(n) = \begin{cases} \textit{recognize}, & \text{if } \cdots \\ \textit{absent}, & \text{otherwise} \end{cases}$$

   Answer $\boxed{(x(n-3), x(n-2), x(n-1), x(n)) = (1,1,1,1)}$

   (b) Determine whether the non-deterministic machine B simulates A and write down the relevant simulation relation if it does.



   Answer Yes, B simulates A with the simulation relation

$$\boxed{\{(start, start), (1, maybe), (11, maybe), (111, maybe)\}}$$

2. **20 points. 5 points for (a), (b), 10 points for (c)** The input signal x and output signal y of an LTI system are related by the differential equation

$$\forall t \in \textit{Reals}, \quad \dot{y}(t) + y(t) = x(t).$$

(a) The frequency response of this system is

$$\forall \omega \in \textit{Reals}, \quad H(\omega) = \boxed{\frac{1}{1+i\omega}}$$

and the *magnitude* and *phase* response for $\omega = 0, \pm 1$, and $\omega \to \pm\infty$ are:

$$\boxed{|H(0)| = 1, \angle H(0) = 0; \quad |H(\pm 1)| = \frac{1}{\sqrt{2}}, \angle H(\pm 1) = \mp\frac{\pi}{4}}$$

$$\boxed{\lim_{\omega\to\pm\infty} |H(\omega)| = 0, \lim_{\omega\to\pm\infty} \angle H(\omega) = \mp\frac{\pi}{2},}$$

(b) Using the hint, the impulse response of this system is

$$\forall t \in \textit{Reals} \quad \boxed{h(t) = \begin{cases} 0, & \text{if } t < 0 \\ e^{-t}, & \text{if } t > 0 \end{cases}}$$

(c) Now consider an LTI system whose impulse response $g = h * h$, where h is as in (2b). Let G be the frequency response of this system. Then

$$\forall \omega \in \textit{Reals}, \quad G(\omega) = [H(\omega)]^2 = \boxed{\frac{1}{(1+i\omega)^2}},$$

$$|G(1)| = \boxed{\frac{1}{2}}, \quad \angle G(1) = \boxed{-\frac{\pi}{2}}$$

$$\forall t \in \textit{Reals}, \quad g(t) = \boxed{(h * h)(t)} = \boxed{\begin{cases} 0, & \text{if } t < 0 \\ \int_0^t e^{-\tau} e^{-(t-\tau)} d\tau = te^{-t}, & \text{if } t > 0 \end{cases}}$$

3. **20 points, 4 points each part** Let M be a state machine with input and output alphabet $\{0, 1, absent\}$. State whether the following propositions are true or false.

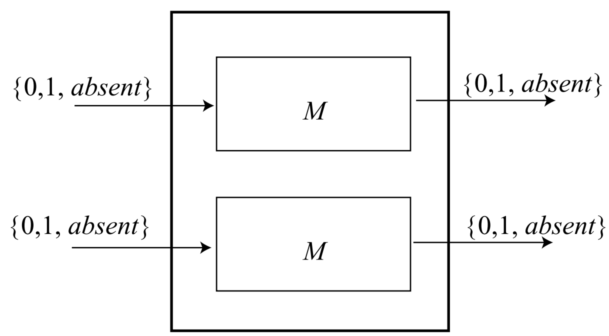
   (a) Suppose M has a *finite* number of states. Let $y = (y(0), y(1), \cdots,)$ be the output signal corresponding to the input signal $x = (0, 0, 0, \cdots)$ (all zero sequence). Then the output signal y must be eventually periodic, i.e. there are integers N, p such that $\forall n > N, y(n+p) = y(n)$.
   Answer: $\boxed{\text{True}}$

   (b) Suppose the output signal y of M is related to its input signal x by: $\forall n \geq 0$,

$$y(n) = \begin{cases} 1, & \text{if } x(0), \cdots, x(n) \text{ contain an } unequal \text{ number of 0s and 1s,} \\ 0, & \text{otherwise} \end{cases}$$

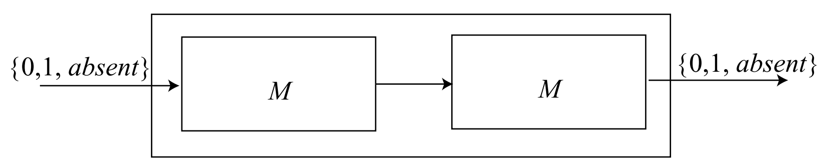
   Then M has an *infinite* number of states.
   Answer: $\boxed{\text{True}}$

   (c) Suppose all the states of M are *reachable* (from the initial state). Then all states of the side-by-side composition of M with itself are reachable. The composition is shown below.



   Answer: $\boxed{\text{True}}$

   (d) Suppose all the states of M are *reachable* (from the initial state). Then all states of the cascade composition of M with itself are reachable. The composition is shown below.



   Answer: $\boxed{\text{False}}$

   (e) Suppose N is another deterministic state machine that simulates M. Then M simulates N. **Answer**: $\boxed{\text{True}}$

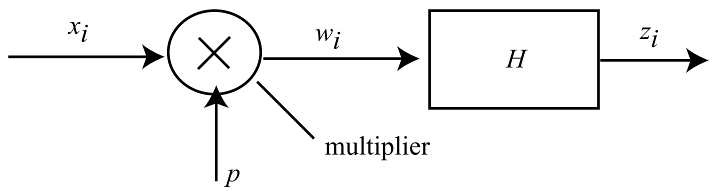
4. **25 points, 5 points each part** This is a block diagram of a sampling and reconstruction system. The input signal x_i : *Reals* $\to$ *Complex* is multiplied by the periodic impulse train p to produce the sampled signal w_i. Here

$$\forall t \in \textit{Reals}, \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

T is the sampling period ($f = T^{-1}$ is the sampling frequency in Hz). The ideal reconstruction filter has frequency response H given by

$$\forall \omega, \quad H(\omega) = \begin{cases} T, & \text{if } |\omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

Assume below that $f = 8,000$ Hz, $T = 125\mu$s.



(a) The Fourier transform of p is

$$\forall \omega \in \textit{Reals}, \quad P(\omega) = \boxed{\tfrac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \tfrac{2\pi}{T} k)}$$

(b) In terms of X_i, the Fourier Transform of x_i, the Fourier Transform of w_i is

$$\forall \omega \in \textit{Reals}, \quad W_i(\omega) = \boxed{\tfrac{1}{T} \sum_{k=-\infty}^{\infty} X_i(\omega - \tfrac{2\pi}{T} k)}$$

and the Fourier Transform of z_i is

$$\forall \omega \in \textit{Reals}, \quad Z_i(\omega) = \boxed{\sum_{k=-\infty}^{\infty} X_i(\omega - \tfrac{2\pi}{T} k), |\omega| < \tfrac{\pi}{T}; = 0, \text{ else}}$$

(c) Suppose $\forall t$, $x_1(t) = \cos(2\pi \times 1000t)$. Then $\forall \omega \in$ *Reals*,

$$\begin{aligned} X_1(\omega) &= \boxed{\pi[\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)]} \\ W_1(\omega) &= \boxed{\tfrac{\pi}{T} \sum_k [\delta(\omega - 2000\pi - 16000\pi k) + \delta(\omega + 2000\pi - 16000\pi k)]} \\ Z_1(\omega) &= \boxed{\pi[\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)] = X_1(\omega)} \end{aligned}$$

(d) Suppose $\forall t$, $x_2(t) = \cos(2\pi \times 7000t)$. Then $\forall \omega \in$ *Reals*,

$$\begin{aligned} X_2(\omega) &= \boxed{\pi[\delta(\omega - 14000\pi) + \delta(\omega + 14000\pi)]} \\ W_2(\omega) &= \boxed{\tfrac{\pi}{T} \sum_k [\delta(\omega - 14000\pi - 16000\pi k) + \delta(\omega + 14000\pi - 16000\pi k)]} \\ Z_2(\omega) &= \boxed{\pi[\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)] = X_1(\omega)} \end{aligned}$$

(e) Suppose $\forall t$, $x_3(t) = \cos(2\pi \times 1000t) - \cos(2\pi \times 7000t)$. Then, $\boxed{\text{by linearity}}$, $\forall \omega \in$ *Reals*,

$$Z_3(\omega) = \boxed{Z_1(\omega) - Z_2(\omega) = 0}$$

and so $\forall t \in$ *Reals*,

$$\boxed{z_3(t) = 0}$$

5. **20 points, 5 points each part** The step input for a continuous time system is defined as $x(t) = 0, t < 0; x(t) = 1, t \geq 0$, and for a discrete time system it is defined as $x(n) = 0, n < 0; x(n) = 1, n \geq 0$.

   (a) If the impulse response of a continuous time LTI system is

   $$\forall t, \quad h(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & t \geq 0 \end{cases}$$

   its step response is

   $$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \boxed{\begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & t \geq 0 \end{cases}}$$

   (b) If the impulse response of a continuous time LTI system is

   $$h(t) = \begin{cases} e^{-|t|}, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

   its step response is

   $$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \boxed{\begin{cases} e^{t}, & t < 0 \\ 1, & t \geq 0 \end{cases}}$$

   (c) If the impulse response of a discrete time LTI system is

   $$h(n) = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

   its step response is

   $$s(n) = \sum_{k=-\infty}^{n} h(k) = \boxed{\begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 2, & n = 1 \\ 3, & n = 2 \\ 3, & n > 2 \end{cases}}$$

   (d) If the impulse response of a discrete time LTI system is

   $$h(n) = \begin{cases} 1, & n = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

   its step response is

   $$s(n) = \sum_{k=-\infty}^{n} h(k) = \boxed{\begin{cases} 0, & n < 1 \\ 1, & n = 1 \\ 2, & n = 2 \\ 3, & n = 3 \\ 3, & n > 3 \end{cases}}$$

6. **10 points** For each continuous time signals x_i, write down its Fourier transform X_i

(a) $\forall t, x_1(t) = e^{i20t}, \quad \forall \omega, X_1(\omega) = \boxed{2\pi\delta(\omega - 20)}$

(b) $\forall t, x_2(t) = 1, |t| < T; x_2(t) = 0, |t| > T.$

$$\forall \omega, X_2(\omega) = \boxed{\frac{2\sin(T\omega)}{\omega}}$$

(c) $\forall t, x_3(t) = x_1(t) \times x_2(t).$

$$\forall \omega, X_3(\omega) = \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\Omega)X_2(\omega - \Omega)d\Omega = \frac{2\sin(T(\omega-20))}{\omega-20}}$$

(d) The unit of ω above is $\boxed{\text{rad/sec}}$

7. **10 points** For each of the following discrete-time systems with input signal x and output signal y, state whether it is linear (L), time-invariant (T), linear and time-invariant (LTI), or none (N).

$\forall n, \quad y(n) = x(2-n)$ **Answer: L**

$\forall n, \quad y(n) = [x(n-1)]^2$ **Answer: TI**

$\forall n, \quad y(n) = \sum_{m=-\infty}^{\infty} 0.5^{|m|} x(n-m)$ **Answer: LTI**

$\forall n, \quad y(n) = x(2-n) + x(n-2)$ **Answer: L**

$\forall n, \quad y(n) = n^2 x(n)$ **Answer: L**