Try to keep your answers succinct. The exam is CLOSED BOOK. All questions count equally. First, a few helpful theorems and definitions. Just because a theorem is mentioned, it may not be helpful on the exam.

**Lemma:** The Pumping Lemma

If $L$ is regular

then $\exists n(\forall z \in L \text{ such that } |z| \geq n)(\exists uw \text{ such that } z = uwv \text{ and } |u| \leq n \text{ and } |v| \geq 1)(\forall i) : u^iw \in L$

**Lemma:** The contrapositive of the Pumping Lemma

If $(\forall n)(\exists z \in L \text{ such that } |z| \geq n)(\forall uvw \text{ such that } z = uvw \text{ and } |uw| \leq n \text{ and } |v| \geq 1)(\exists i) : u^iw \notin L$

then $L$ is not regular.

**Theorem:** Rice’s theorems: Let $L_P$ be the set of machines with property $P$. If $P$ is non-trivial, $L_P$ is undecidable. Further, $L_P$ is r.e. if and only if $P$ satisfies the following three conditions:

1. If $L \in P$ and $L \subseteq L'$ for some r.e. $L'$, then $L' \in P$.
2. If $L$ is an infinite language in $P$, then there exists a finite subset of $L$ in $P$.
3. The set of finite languages in $P$ is enumerable.

**3-SATISFIABILITY (3SAT)**

**INSTANCE:** A boolean formula, $F$, which is an AND of clauses where each clause is an OR of 3 literals.

**QUESTION:** Is $F$ satisfiable?

**3-DIMENSIONAL MATCHING (3DM)**

**INSTANCE:** A set $M \subseteq W \times X \times Y$, where $|W| = |X| = |Y| = q$ are disjoint sets.

**QUESTION:** Does $M$ contain a matching, $M' \subseteq M$, such that no two elements of $M'$ agree in any coordinate.

**VERTEX COVER (VC)**

**INSTANCE:** A graph $G$ and integer $K$

**QUESTION:** Is there a subset of $K$ vertices which cover all the edges?

**CLIQUE**

**INSTANCE:** A graph $G$ and integer $K$

**QUESTION:** Does the graph contain a clique (completely connected subgraph) of $K$ vertices?

**HAMILTONIAN CIRCUIT (HC)**

**INSTANCE:** A graph $G$

**QUESTION:** Is there a cycle through all the vertices of $G$

**PARTITION**

**INSTANCE:** A finite set $A$ and a “size” $s(a) \in \mathbb{Z}^+$ for each $a \in A$.

**QUESTION:** Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$$
1. Prove or disprove the following languages are regular:
   
   (a) \( L_a = \{ a^t b^s : s \geq t \geq 1 \} \).
   
   (b) \( L_b = \{ a^t b^s : t > s \geq 1 \} \). For the proof, use set closure properties and your result about \( L_a \). No credit for using the pumping lemma.
   
   (c) \( L_c = \{ w : w \text{ contains the substring } 0011 \} \)

2. Which of the following are r.e.? Give a proof. (Hint: Any reductions can be done from \( L_a \) by creating an \( M' \) from \( \langle M, w \rangle \) which accepts either \( \emptyset \) or \( \Sigma^* \) depending on whether \( M(w) \) rejects or accepts.)
   
   (a) \( L_{3M} = \{ \langle M_1, M_2, M_3 \rangle : \text{At least two of the machines accept the same language.} \} \)
   
   (b) \( L_{\overline{3M}} \)
   
   (c) \( L = \{ \langle M \rangle : M(e) \text{ never moves past the } |Q|^\text{th} \text{ tape square}. \text{ (}Q\text{ is the set of states of } M.\text{)} \} \)

3. Of the following three problems, prove one is in NP, prove one in co-NP, and prove the third is in P.
   
   (a) INSTANCE: Two graphs on the same vertex set \( G = (V, E) \) and \( H = (V, E') \).
       QUESTION: Are \( G \) and \( H \) non-isomorphic?
       (Note that it says “non-isomorphic” rather than “isomorphic”.)

   (b) INSTANCE: A boolean formula, \( F \), on the 100 variables \( \{x_1, \ldots, x_{100}\} \).
       QUESTION: Is \( F \) unsatisfiable?

   (c) INSTANCE: A binary number \( n > 1 \) in binary.
       QUESTION: Is \( n \) composite? (“Composite” means “not prime”).

4. Prove FEEDBACK VERTEX SET is NP-complete.
   
   FEEDBACK VERTEX SET
   
   INSTANCE: Directed graph \( G = (V, E) \) and integer \( K \).
   
   QUESTION: Is there a subset \( V' \subset V \) such that \( |V'| \leq K \) and every directed circuit in \( G \) includes at least one vertex from \( V' \).

5. Prove HITTING STRING is NP-complete:
   
   INSTANCE: An integer \( n \) and a set of strings \( A \subset \{0,1,\#\}^n \).
   
   QUESTION: Is there a string \( x \in \{0,1\}^n \) such that for each string \( a \in A \) there is some \( i, 1 \leq i \leq n \), for which the \( i^{th} \) symbol of \( a \) and the \( i^{th} \) symbol of \( x \) are identical.

   For example, \( A = \{11\#0,0###\#,\#0\#,\#\#\#1,0\#1\#\} \) is a positive instance by choosing \( x = 0101 \).