1. (12 pts.) Short-answer questions

Translate each of the following claims into symbolic form. For instance, a good translation of “n is either at least three or at most five” would be “n ≥ 3 \land n ≤ 5.”

Then, state whether the claim is true or false, and briefly justify your answer.

(a) [3 pts.] There is some natural number whose square root is not a natural number.

(b) [4 pts.] For every natural number \( n \), one can find another natural number \( m \) that is strictly smaller than \( n \).

(c) [5 pts.] For each natural number \( k \) there is some lower bound \( \ell \) so that \( k^\ell \geq n! \) when \( n \geq \ell \).
2. (12 pts.) Reachability

In chess, a bishop can move diagonally in any of the four directions. Consider a 3×3 board, with a bishop initially placed at the location marked ‘B’ (see below). Prove that it can never reach the square marked ‘X’.

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B

X
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3. (16 pts.) Proof by induction

Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation

$$a_n = 2a_{n\lceil \frac{n}{2} \rceil} - a_{n\lceil \frac{n}{2} \rceil - 2}$$

for $n \geq 2$ and $a_0 = 1, a_1 = 2$.

Consider the following argument:

**Theorem 1** $a_n \leq n + 2$ for all $n \geq 0$.

**Proof:** We use strong induction on $n$. The base cases $n = 0$ and $n = 1$ hold, since $a_0 = 1 \leq 0 + 2$ and $a_1 = 2 \leq 1 + 2$. Now if $a_i \leq i + 2$ for each $i = 0, 1, \ldots, n-1$, for some $n \geq 2$, then we have

$$a_n = 2a_{n\lceil \frac{n}{2} \rceil} - a_{n\lceil \frac{n}{2} \rceil - 2} \leq 2(n \lceil \frac{n}{2} \rceil + 2) - (n \lceil \frac{n}{2} \rceil - 2) \leq 2n + 2 \leq n + 2,$$

which shows that $a_n \leq n + 2$ holds for all $n \geq 0$. □

(a) [6 pts.] Critique the above proof.

(b) [10 pts.] Give a better proof of the theorem.
4. (10 pts.) Matchings

Recall that a matching on \( n \) boys and \( m \) girls is a pairing where each boy is married to exactly one girl and each girl is married to exactly one boy.

(c) [5 pts.] Let \( M \) be a stable matching on \( n \) boys and \( n \) girls where Alice is paired with Bob. Now Alice and Bob fly off the Bermuda on vacation. We are left with a matching, call it \( L \), on the remaining \( n-1 \) boys and \( n-1 \) girls according to who is still paired up. Is \( L \) guaranteed to be a stable matching, if \( M \) is stable? Prove your answer.

(d) [5 pts.] If \( M, M' \) are two matchings, let \( M \sqcup M' \) denote the configuration where each girl is married to the better of her two partners in \( M \) and \( M' \) (according to that girl’s preference list). Is \( M \sqcup M' \) guaranteed to be a matching? Prove your answer.
(Note that none of the matchings here are required to be stable.)

Finished! You’re done; this is the last page; there are no more questions.