
Midterm Exam 1 Solutions

CS70 Blum/Wagner, 9 March 2001

This is a CLOSED BOOK examination. One page of notes is permitted. Calculators are permitted. Do all your work on the pages of this examination. Give reasons for all your answers.

Problem 1 (Short answer grab-bag) [10 Points]
For parts (a)-(c), justify your answer briefly.
(a) [2 Points] True or False: (P => Q) => (Q => P) always holds, for all propositions P, Q.

> **False. Counter example: P=False, Q=True. This is a converse error.**

(b) [3 Points] True or False: ((P v Q) => Q) => (Q => (P v Q)) always holds, for all propositions P, Q.

> **True. (Q => (P v Q)) = True for all P,Q, and (anything => True) is true.**
> **(Don't be fooled into thinking this is another converse error. It's not.)**
> **(Common error: Not realizing that (False => False) is True.)**

(c) [3 Points] Recall that two sest S,T are said to be *disjoint* if $S \cap T$ = Ø. Can two events A and B be simultaneously independent and disjoint?

> **Yes. An example: If A = Ø, then A isect B = Ø and Pr[A]=0.**
> **So, Pr[A isect B] = 0 = 0 × Pr[B] = Pr[A]Pr[B], and A,B are independent and disjoint.**

(d) [2 Points] Recall that two random variables X and Y are *independent* just if the pair of events X=*i* and Y=*j* are independent no matter how you choose the values *i* and *j*. Which of the following most accurately expresses the proposition that X and Y are *not* independent? (You do not need to justify your answer.)

(i) $\forall i. \forall j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i]\Pr[Y = j]$

(ii) $\forall i. \exists j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i]\Pr[Y = j]$

(iii) $\exists i. \forall j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i]\Pr[Y = j]$

(iv) $\exists i. \exists j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i]\Pr[Y = j]$

The answer is (iv) since $not(\forall i.P(i)) = \exists i.not(P(i))$.

Problem 2 (Permutations) [25 Points]
Let *S* be the sample space of all permutations on the *n* letters {1,2,...,*n*}, with the uniform probability distribution. In each of the following, give reasons for all your answers. (You may find the next page, which has been left blank, useful for justifying your answers.)

(a) [1 Point] The uniform probability distribution assigns to every permutation on *n* letters the probability: 1/(n!)

Each of the n! permutations is assigned the same probability, and these probabilities must sum to 1.

Define the random variable X_i to be the number of cycles of length *i*, i.e., X_i maps each permutation to an integer equal to the number of cycles in that permutation that are of length *i*.

(b) [1 Point] For the permutation (124)(36)(5)(7) on *n*=7 letters:

X_1= 2 , X_2= 1 , X_3= 1 , X_4= 0 .

2 cycles of length 1: (5) and (7), 1 of length 2: (56), 1 of length 3: (124)

For general positive integers *n*, give:

(c) [3 Points] $E[X_1]$ = 1 .

Define random variable $X_{1i} = \begin{cases} 1 & if\ i \to i \\ 0 & otherwise \end{cases}$

Note that $X_1 = \sum_{i=1}^{n} X_{1i}$

By linearity of expectation, $E[X_1] = \sum_{i=1}^{n} E[X_{1i}] = \sum_{i=1}^{n} \Pr[i \to i] = \sum_{i=1}^{n} \frac{1}{n} = 1$

(d) [7 Points] $E[X_2]$ = 1/2 .

Define random variable $X_{2i} = \begin{cases} 1/2 & \text{if } i \leftrightarrow j \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$

Note that $E[X_2] = \sum_{i=1}^{n} E[X_{2i}]$

By Linearity of expectation, $E[X_2] = \sum_{i=1}^{n} E[X_{2i}]$

$$Now,\ E[X_{2i}] = \Pr\left[X_{2i} = \frac{1}{2}\right]\frac{1}{2} = \frac{n-1}{n}\cdot\frac{1}{n-1}\cdot\frac{1}{2} = \frac{1}{2n}$$

$$So,\ E[X_2] = \sum_{i=1}^{n}\frac{1}{2n} = \frac{1}{2}$$

For an integer $k \in \{1,2,...,n\}$, what is

(e) [8 Points] $E[X_k]$ = ____1/k_____.

Define random variable $X_{2i} = \begin{cases} 1/k & \text{if } i \text{ participates in a cycle of length } k \\ 0 & \text{otherwise} \end{cases}$

$$E[X_{ki}] = \frac{1}{k}\Pr\left[X_{ki} = \frac{1}{k}\right] = \frac{1}{k}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdots\frac{1}{n-k+1} = \frac{1}{kn}$$

$$E[X_k] = \sum_{i=1}^{n} E[X_{ki}] = \sum_{i=1}^{n}\frac{1}{kn} = \frac{1}{k}$$

Define the random variable X to be the number of cycles in a permutation. In other words, X maps any permutation to a positive integer equal to the number of cycles in that permutation.

(f) [1 Point] For the permutation (124)(36)(5)(7), X = ____4_____.

The 4 cycles are (124), (36), (5), and (7).

(g) [1 Point] Is $X = X_1 + X_2 + ... + X_n$? ___Yes_____.

X = # of cycles = (# of cycles of length 1) + (# of cycles of length 2) + ... = $X_1+X_2+\dots+X_n$

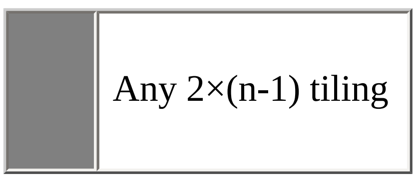
(h) [3 Points] E[X] = ___ $H_n = \sum_{k=1}^{n} \frac{1}{k}$ ___.

E[X] = $E[X_1] + E[X_2] + \dots + E[X_n] = 1 + 1/2 + \dots + 1/n = H_n$

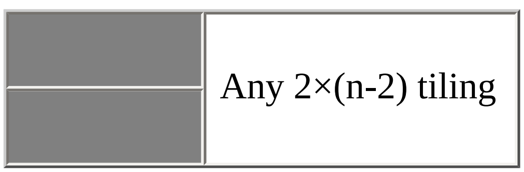
Problem 3 (Tiling) [15 Points]
Let D_n be the number of ways to tile a $2\times n$ checkerboard with dominos, where a domino is a 1×2 piece.

Prove that $D_n \le 2^n$ for all positive integers n. (Hint: find a recurrence relation.)

Note that every 2×n tiling takes one of the following two forms:

(a) 

One vertical domino on left.

(b) 

Two horizontal, stacked dominos on left.

These two cases are mutually exclusive and exhaust the possibilities, and there are D_{n-1} **configurations of form (a) and** D_{n-2} **of (b).**

Therefore, $D_n = D_{n-1} + D_{n-2}$ **for** $n \ge 2$, $D_0 = D_1 = 1$

Claim: $D_n \le 2^n$ **for all** $n \ge 0$**;**

Proof: By strong induction on n.

Base Case: $D_0 = 1 \le 2^0$ and $D_1 = 1 \le 2^1$, so the claim is true for n=0 and n=1.

Inductive Step: We show that $\forall k \geq 2. \left(D_0 \leq 2^0 \wedge \cdots \wedge D_{k-1} \leq 2^{k-1}\right) \Rightarrow D_k \leq 2^k$

Assume $D_0 \leq 2^0 \wedge \cdots \wedge D_{k-1} \leq 2^{k-1}$, then

$$\begin{aligned} D_k &= D_{k-1} + D_{k-2} && \textit{by recurrence relation} \\ &\leq 2^{k-1} + 2^{k-2} && \textit{by inductive hypothesis} \\ &= \left(\frac{1}{2} + \frac{1}{4}\right) 2^k \\ &\leq 2^k && \textit{Q.E.D.} \end{aligned}$$
