I ACCEPT NO RESPONSIBILITY FOR ERRORS ON THIS SHEET. I assume that $E = \Omega(V)$.

Data structures

- Binary heaps are implemented using a heap-ordered balanced binary tree. Binomial heaps use a collection of B_k trees, each of size 2^k. Fibonacci heaps use a collection of trees with properties a bit like B_k trees. (The operation HEAPIFY below makes a heap with n elements without doing n INSERTs.)

Procedure	Binary heap (worst case)	Binomial heap (worst case)	Fibonacci heap (amortized)
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$
HEAPIFY	$O(n)$	$O(n)$	$O(n)$
INSERT	$O(\lg n)$	$O(\lg n)$	$O(1)$
MINIMUM	$O(1)$	$O(\lg n)$	$O(1)$
EXTRACT-MIN	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$
UNION	$O(n)$	$O(\lg n)$	$O(1)$
DECREASE-KEY	$O(\lg n)$	$O(\lg n)$	$O(1)$
DELETE	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

- Binary search trees implement all the operations of heaps (except UNION) in addition to SEARCH. Virtually all the operations take time $\Theta(\log n)$.
- The union-find data structure implements the operations UNION(x, y, `label`) and `label`←FIND(x) on a collection of disjoint sets. Initially (before any UNION operation) each of n elements is in its own set of size one. A total of m disjoint set operations take time $O(m\alpha(m,n))$, where $\alpha(m,n)$ is a ridiculously slowly growing function. $\alpha(m,n)$ is $o(\log^*(m))$, $o(\log m)$, and $\Omega(1)$.

Sorting

- For comparison-based sorting algorithms, Heapsort and mergesort take time $O(n \log n)$, and quicksort takes expected time $O(n \log n)$. An information theoretic lower bound for any comparison based sort is $\log(n!) = \Omega(n \log n)$.
- For n numbers known to fall within a range $\{1, \ldots, N\}$, radix sort will take time $O(n+N)$. Linear time algorithms are often possible if more can be known about the numbers besides how to pairwise compare them.
- Order statistics (the k^{th} largest element of n elements, or the median of n elements) can be found in time $\Theta(n)$ by a comparison based algorithm. The algorithm chooses a pivot by recursively computing the median of the medians of $n/5$ subsets of 5 elements each. Once all elements are compared to the pivot, 1/4 of the elements can be discarded, as they are all known to be greater than (or, less than) the k^{th} largest element.

Exploring graphs

- Breadth first search (BFS) takes $O(E)$ time and finds shortest paths.
- Depth first search (DFS) takes $O(E)$ time. Also in this time you can have preorder numberings (or discover times), postorder numberings (or finish times), classification of edges as forward, back, back-cross or back-cross-tree edges.
- A topological sort of a dag can be found in $O(E)$ time by reversing the postorder numbers in a DFS.
- Strongly connected components (SCC's) can be found in $O(E)$ time. Note that the component graph, G^{SCC} is acyclic, so you can topologically sort it.

Minimum Spanning Trees (MST's)

- If $A \subset E$ is *part of* a MST, and S is a cut which no edge in A crosses, then the minimum edge across the cut can be added to A to yield part of a MST.
- Kruskal's grows a collection of trees by always adding the cheapest edge which connects two trees, taking time $O(E \lg V)$ to sort the edges.
- Prim's algorithm grows a single tree from a vertex by always adding the cheapest edge out from the tree, taking time $O(E + V \lg V)$ if a Fibonacci heap is used.

Shortest path problems

- Single-source shortest paths for non-negative edge weights can be found by Dijstra's algorithm. Like Prim's, grow a tree, always adding the vertex with the cheapest path from the source by extending the tree by only one edge. Takes $O(E + V \lg V)$ using a Fibonacci heap.
- Single-source shortest paths for edge weights which may be negative can be found using Bellman-Ford. Make V passes over the graph, updating shortest path estimates to each vertex *relaxing edges*. Either a negative cycle will be found or a shortest path tree in time $O(EV)$.
- All-pairs shortest path problems had a few algorithms:
  - Matrix-multiply like algorithms taking time $O(V^4)$ or $O(V^3 \log V)$.
  - Floyd Warshall takes time time $O(V^3)$. They use dynamic programming to solve

    $$d_{ij}^{(k)} = \text{the shortest path from } v_i \text{ to } v_j \text{ using paths going } \textit{through } \{v_1, \ldots, v_k\}$$

  - Johnson's algorithm first solves a single source shortest path problem from an added vertex, s, (with edges (s, v), $v \in V$ of weight 0) to reweight the edges by:

    $$\hat{w} = w(u, v) + \delta(s, u) - \delta(s, v)$$

    This results in positive edge weights, and Bellman-Ford can be used to take time $O(EV)$.

Linear programming

- In linear programming, the goal is to optimize a linear objective function subject to linear inequality constraints. No algorithm were discussed, but polynomial time algorithms exist for linear programming.
- In integer linear programming, the goal is to find an integer solution to a linear programming problem. No polynomial time algorithm is known nor is likely to exist for this NP-complete variant.

Flow networks and maximum flows

- Capacities satisfy $c(u, v) \geq 0$. Flows satisfy

  Capacity constraints : $f(u, v) \leq c(u, v)$
  Skew symmetry: $f(u, v) = -f(v, u)$
  Flow conservation: $\forall u \in V - \{s, t\} : \sum_{u \in V} f(u, v) = 0$

- The residual capacities are given by $c_f(u, v) = c(u, v) - f(u, v)$.
- The min-cut max-flow theorem proves the maximum flow is equal to the minimum capacity over all cuts. Further, if there are no augmenting paths in the residual graph, a maximum flow has been obtained.
- Ford-Fulkerson finds paths from s to t in the residual graph to augment the flows until no more paths can be found, taking time $O(Ef^*)$, where f^* is the value of the max-flow.

- Edmonds-Karp improves on this by always choosing the shortest augmenting path (i.e., fewest edges), finding the max-flow in time $O(VE^2)$.

Number theoretic algorithms Throughout, define β to be the length of bits in all the numbers involved.

- The greatest common divisor $\gcd(a,b) = \gcd(b, a \bmod b)$, yielding Euclid's algorithm taking $O(\beta)$ arithmetic operations.
- If $\gcd(a,b) = d$ then there is an x and y so that $ax + by = d$. Euclid's algorithm can be adjusted to calculated x and y efficiently.
- (Chinese Remainder Theorem) If $\gcd(n_1, n_2) = 1$ and $n = n_1 n_2$, then there is a one-to-one mapping between numbers $a \bmod n$ and pairs $(a_1 mod n_1, a_2 mod n_2)$ so that $a_i = a \bmod n_i$. To compute a, find x and y so that $n_1 x + n_2 y = 1$ and notice that $(1,0)$ maps to $n_2 y \bmod n$ and $(0,1)$ maps to $n_1 x \bmod n$. So, (a_1, a_2) maps to $a_1 n_2 y + a_2 n_1 x \bmod n$.
- (Fermat's Little Theorem) For p prime, $1 \leq a < p$, $a^{p-1} \equiv 1 \pmod{p}$.
- A pseudo-prime test is to check if $2^{n-1} \equiv 1 \pmod{n}$; output "prime?" if yes, "composite!" if no. Very few composites look like primes. A randomized primality test chooses k random values of a in the range $1 \leq a < n$. For each, calculate if $a^{n-1} \equiv \pm 1 \pmod{n}$. If one is not ± 1 output "composite!". If all are 1 output "composite?". Otherwise output "prime?". This test fails with probability $\leq \frac{1}{2^k}$.
- In the *RSA public-key cryptosystem,* a participant creates her public and private keys with the following precedure.
  1. Select at random two large prime numbers p and q.
  2. Compute n by the equations $n = pq$.
  3. Select a small odd integer e that is relatively prime to $\phi(n) = (p-1)(q-1)$.
  4. Compute d as the multiplicative inverse of $e \bmod \phi(n)$.
  5. Publish the pair $P = (e, n)$ as her *RSA public key.*
  6. Keep secret the pair $S = (d, n)$ as her *RSA secret key.*

  To encode message M, computer $M^e \bmod n$. To decode cybertext C, compute $C^d \bmod n$

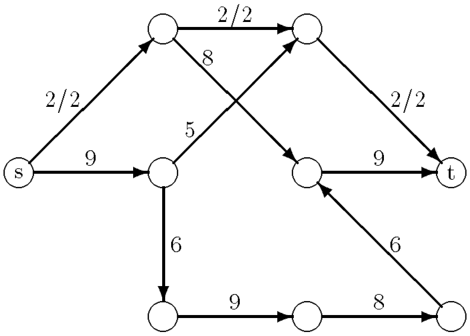
CS-170 Exam 3 April 26, 1995
David Wolfe

You should not need to write any code for this exam. Please make your answers as brief and as clear as possible. I highly recommend crossing out mistakes with a few dark strokes of the pen rather than erasing the work completely in case it is worth partial credit.

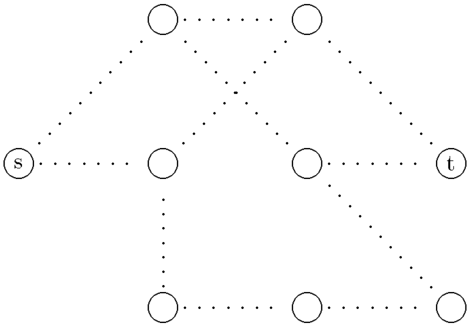
Below is a summary line for each question on the exam. You may use the backs of pages if you need more space, but please indicate for the grader you've done so: For example, "Continued on back of page 4". Please leave the exam stapled. You can look pick up a solution set (with the questions repeated) when you leave.

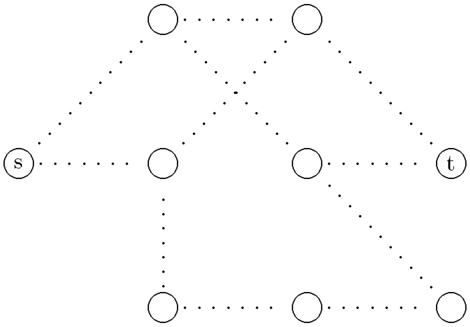
1.	Edmonds-Karp algorithm	/20
2.	Linear programming definitions	/15
3.	Job scheduling linear program	/30
4.	Non-cycle edges	/20
5.	Unit-capacity edges	/30
Total	(Extra points possible)	/115

1. (20 points) The following gives capacities and the flow after executing one round of the Edmonds-Karp improvement to the Ford-Fulkerson algorithm:

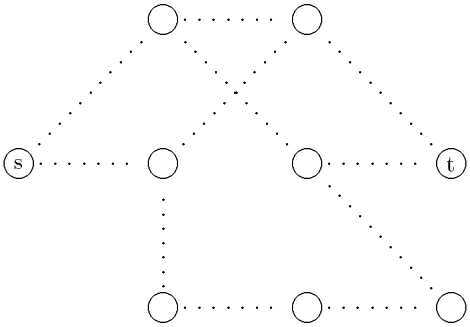


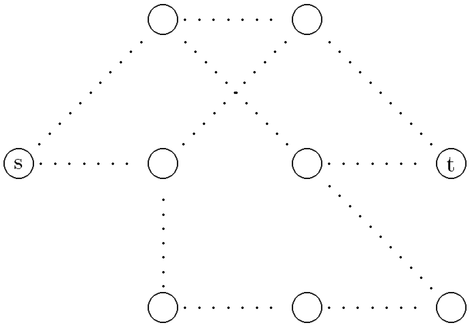
   (a) Draw the residual graph, G_f in the space below. The dotted edges (and the extra graph) are for your convenience. You need not include edges into s or from t in G_f.





   (b) Draw the flow obtained after one more iteration of Edmonds-Karp. Please do not indicate the capacities; you only need to draw edges with flow.





2. (15 points) Consider the five problems labeled (a)-(e) below:

$$\begin{array}{rcl} \min\ 3x_1 + 4x_2 & s.t. & \\ x_1^2 + x_2 & \leq & 5 \\ x_2 & \geq & 0 \end{array}$$

(a)

$$\begin{array}{rcl} \max\ x - 3y + 5z & s.t. & \\ x + y & = & 4 \\ -y + z & \geq & 2 \\ -x + z & \leq & 6 \end{array}$$

(b)

$$\begin{array}{rcl} \min\ x_1 + x_2 & s.t. & \\ x_1, x_2 & \in & \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\} \\ 2x_1 + x_2 & \leq & 5 \\ .5x_1 - 3x_2 & \geq & -4 \end{array}$$

(c)

$$\begin{array}{rcl} \min\ x_1 + x_2 & s.t. & \\ 2x_1 + x_2 & \leq & 5 \\ .5x_1 - 3x_2 & \geq & -4 \end{array}$$

(d)

$$\begin{array}{rcl} \min\ x_1 x_2 & s.t. & \\ 3x_1 + 2x_2 & \geq & 10 \end{array}$$

(e)

Of the five, two are linear programs and one is an integer linear program. Indicate which in the boxes below:

Linear program ☐
Linear program ☐
Integer linear program ☐

3. (30 points) We want to determine the optimal scheduling of m jobs to a machine such that:

   - All jobs must be completed within n weeks.
   - Job i requires a total of r_i hours.
   - At most h_j hours can be scheduled on the machine during week j.
   - There is a cost c_{ij} for each hour that job i is assigned to the machine during week j.

   This problem can be formulated as a linear program with $m \times n$ variables, where x_{ij} is the number of hours the machine spends on job i during week j.

   (a) Write the objective function that minimizes the total cost for a possible schedule.

   (b) For job i which requires r_i hours, write the corresponding linear constraint.

   (c) For week j which has at most h_j hours available, write the constraint corresponding to the jobs scheduled during this week.

   (By the way, the additional constraints, "$\forall i, j:\ x_{ij} \geq 0$," will complete the linear program.)

4. (20 points) Give an $O(E+V)$ algorithm to find all edges in a directed graph $G = (V, E)$ which are not contained in any cycle. Hint: Use depth first search, bread first search, strongly connected components and/or topological sort as subroutines. (If your algorithm is simple and clearly stated, no justification is required. A one sentence solution could receive full credit.)

5. (30 points) Let $G = (V, E)$ be a flow network with source s, sink t, and suppose each edge $e \in E$ has capacity $c(e) = 1$. Assume also, for convenience, that $|E| = \Omega(V)$.

   (a) Suppose we implement the Ford-Fulkerson maximum-flow algorithm by using depth-first search to find augmenting paths in the residual graph. What is the worst case running time of this algorithm on G?

   (b) Suppose a maximum flow for G has been computed, and a new edge with unit capacity is added to E. Describe how the maximum flow can be efficiently updated. (Note: It is not the value of the flow that must be updated, but the flow itself.) Analyze your algorithm.

   (c) (Extra credit) Suppose a maximum flow for G has been computed, but an edge is now removed from E. Describe how the maximum flow can be efficiently updated in $O(E + V)$ time.

MORE SPACE IF REQUIRED