There are 6 problems marked (E), and 4 problems (H). Each question is 10 points, but your two highest scores on a (H) question are doubled. (It is possible to score 120 points.) 90 points is enough for an A on the exam, so a student who gets two (H) questions and 6 of the remaining 8 question has an A with 10 points to spare.

1. (E) Two six sided dice are rolled. For each pair of events in the following table, determine if they are independent and/or disjoint.

<table>
<thead>
<tr>
<th>Event A</th>
<th>Event B</th>
<th>Independent?</th>
<th>Disjoint?</th>
</tr>
</thead>
<tbody>
<tr>
<td>First die comes up 3</td>
<td>First die comes up 3 or 4</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>First die comes up 6</td>
<td>First die comes up 1 or 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First die comes up 6</td>
<td>Second die comes up 1 or 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First die comes up 5</td>
<td>Dice add to 6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First die comes up 5</td>
<td>Dice add to 7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First die comes up 5</td>
<td>Dice add to 12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First die comes up 5</td>
<td>Dice add to 13</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. (E) Prove that all planar embeddings of a given connected planar graph have the same number of faces.

3. (E) A 5 card hand is dealt from a standard 52 card deck. Let the events

$$Q = \text{“The hand contains at least one Queen.”}$$

$$H = \text{“The hand contains at least one Heart.”}$$

Calculate $P\{Q\}$, $P\{H\}$, $P\{Q \lor H\}$ and $P\{Q \land H\}$. (Be sure to calculate the easier of $P\{Q \lor H\}$ and $P\{Q \land H\}$ first!)

4. (E) How many 4-digit campus telephone numbers have one or more consecutive repeated digits? (Each digit is randomly selected from \{0,1,\ldots,9\}. 4422 counts, but 2424 doesn’t.)

5. (E) A tree has $6k$ nodes,
   - $2k$ nodes of degree 1
   - $3k$ nodes of degree 2
   - $k$ nodes of degree 3

Find $k$ and show that it is uniquely determined.

6. (E) An ASCII character is 8 bits. Suppose each character is transmitted along a modem with an extra parity bit which is the exclusive-or of the 8 bits.
   - (a) Describe the set $C$ of 9-bit code words transmitted.
   - (b) Find the hamming distance, $d$, of $C$.
   - (c) How many errors can be detected in the code?
   - (d) How many errors can be corrected in the code?

7. (H) Let $G$ be a random $n \times n$ bipartite graph with each edge included independently with probability $\frac{1}{n}$. Let $N$ be the number of ways to make a perfect matching in $G$. For example, if $G$ is the following graph, $N = 2$, and the two perfect matchings are listed to the right.
• (7 points) What is E\{N\}?
• (3 points) How does E\{N\} compare with P\{N \geq 1\}? What does this say about the probability G has a perfect matching when n \to \infty?

8. (H) A tournament is a directed graph with exactly one edge between every pair of vertices. In other words, to get a tournament, take a complete undirected graph and direct each edge. Show that every tournament has a hamiltonian path.

Hint: One way to begin a proof is:

Let v be any vertex in tournament G. Partition the vertices of G into three sets, \{v\}, S, and T, where S is the set of vertices in G which point to v, and T is the set of vertices which v points to.

9. (H) Assume each switch in the following circuit will be closed (i.e., a connection is made) independently with probability p.

(a) Find the probability that all switches are closed.
(b) Find the probability that x and y are connected.
(c) You do a test and find that x and y are connected. Now what is the probability that all switches are closed?

10. (H)

(a) Find all winning moves in the following Nimstring position.

(b) Draw the corresponding Dots & Boxes position. How many boxes will you get in a well played game from this position?