Math 123, F.Rezakhanlou, **Final Exam** F95

1. *(7 pts)* What is a limit cycle? Give an example of an ODE in the plane such that: "It has a periodic solution that attracts all nearby solutions".

2. *(7 pts)* Consider the ODE $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -x+y \\ -y(1+x^2) \end{bmatrix}$. Find its critical points. Construct a Liapunov function for each critical point that is a sink.

3. *(7 pts)*
   (a) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a function that satisfies $|F(x) - F(y)| \le L|x-y|$. Let x, y be two solutions to $\frac{dx}{dt} = F(x)$. Show that $|x(t) - y(t)| \le e^{L|t|}|x(0) - y(0)|$ for all t.
   (b) Show that if there are t_1, t_2 with $x(t_1) = x(t_2)$, $t_1 \neq t_2$, then $x(t)$ is a periodic solution.

4. *(8 pts)* Solve $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -\alpha x - \beta y \\ -\beta x + \alpha y \end{bmatrix}$. For what values of α, β, is the equation (a) strictly stable?, (b) neutrally stable? What do the solution curves look like?

5. *(7 pts)* Solve $y''' - y'' + y' - y = \sin x$.

6. *(7 pts)* Find a basis of power series solutions for $y'' = x^2 y$. What is the radius of convergence of your series?

7. *(4 pts)* Show that the equation

$$\begin{cases} y' = -\sqrt{|y|} \\ y(0) = 0 \end{cases}$$

   has infinitely many solutions

8. *(3 pts)* Consider the ODE

$$\begin{cases} x_1'(t) = x_3^2 = x_1 x_2 \\ x_2'(t) = x_3^2 - x_1 x_2 \\ x_3'(t) = x_1 x_2 - x_3^2 \end{cases}$$

   Define $H(x_1, x_2, x_3) = x_1 \log x_1 + x + 2 \log x_2 + 2x_3 \log x_3$.
   Suppose (x_1, x_2, x_3) is a positve solution: $x_1(t) > 0$, $x_2(t) > 0$, $x_3(t) > 0$.
   Show that $H(x_1(t), x_2(t), x_3(t)) \le H(x_1(0), x_2(0), x_3(0))$.