1. (10 points) The following statements are equivalent, except for one that is not. Identify it. Give an explanation for your choice.

   Let **A** be an $n \times n$ matrix.

   (a) The columns of **A** do not span $\mathbf{R^n}$.

   (b) The equation $\mathbf{Ax} = \mathbf{b}$ cannot be solved for all **b** in $\mathbf{R^n}$.

   (c) The linear transformation that sends an arbitrary **x** into **Ax** is one-to-one. pivot in every column

   (d) The reduced row-echelon form of **A** has one row consisting of all zeros.

In order for a transformation to be one to one, it must have a pivot in every column. If this is true, then by default (d) is false (since a column with all zeros has no pivot) and a and b would be false because the pivot in every column would mean that the $n \times n$ matrix is invertible and the columns of A by (IMT) would have to span R^n. This means that $Ax = b$ would have to be solvable for all b in R^n.

2. (20 points) Let T be a linear operator mapping $\mathbf{R}^2$ into $\mathbf{R}^3$ defined by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2).$$

$\frac{20}{20}$

Write down the standard matrix representing T. Is the linear transformation one-to-one? Is it onto $\mathbf{R}^3$?

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

standard matrix = $\begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 16 \\ 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 16 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 \\ 0 & 16 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The linear transformation is one-to-one because there is a pivot in every column. However, the linear transformation is not onto $\mathbb{R}^3$ because there is not a pivot in every row.

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3. (25 points) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

with reduced row-echelon form given by

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

consider the corresponding linear transformation and find a basis for its nullspace and range. For those column vectors that are not in the proposed basis, show that they belong to the appropriate span. −8

$x_1 - x_3 - 2x_4 = 0 \qquad x_1 = x_3 + 2x_4$

$x_2 + 2x_3 + 3x_4 = 0 \qquad x_2 = -2x_3 - 3x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Null Space $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

~~range~~ $= \cancel{\left\{ x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, x_3, x_4 \in \mathbb{R}^2 \right\}}$ −7

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. (25 points)

Given the three vectors $\mathbf{v_1} = (1, -3, 2)$, $\mathbf{v_2} = (2, -6, h)$ and $\mathbf{v_3} = (5, -7, 0)$ in $\mathbf{R}^3$

a) For what values of h is $\mathbf{v_3}$ in the span of the other two vectors?

b) For what values of h is the set of three vectors a linearly independent set?

a) $\begin{bmatrix} 1 & 2 & 5 \\ -3 & -6 & -7 \\ 2 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 8 \\ 0 & h-4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & h-4 & -10 \\ 0 & 0 & 8 \end{bmatrix}$

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$8 \neq 0$

$h=4$ In order for v_3 to be in the span of $v_1 + v_2$, there must be a linear combination such that $x_1v_1 + x_2v_2 = v_3$. Since the [illegible] [illegible] [illegible] [illegible], if $h=4$ is not 0; thus $h=4$.

b) $h-4 \neq 0$ A linear independent set occurs when the only solution is the trivial solution. Since $h-4$ must be a pivot for this

$h \neq 4$ to be true, $h-4 \neq 0$; thus $h \neq 4$.

B

5. (20 points)

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Let the matrix A be given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 14 \end{pmatrix}$$

with an **LU** factorization given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Put $\mathbf{b}=[3,2,1]$. Solve the equation $\mathbf{Ax}=\mathbf{b}$ by solving two linear systems involving triangular matrices.

$LUx=b \quad Ux=y \quad Ly=b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 2 & 1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$$Ux=y \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 11 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 11 \\ -4 \\ 0 \end{bmatrix}$$