FINAL

Introduction to Complex Analysis 185H, Spring 2001: Egilsson

MONDAY, MAY 14, 2001 from 12:30 to 3:30 in room F320 HAAS

Name:________________________________

1 (1/6) Let $f : \mathbf{C} \to \mathbf{C}$ be a holomorphic nonconstant function on $\mathbf{C}$ and define the real valued function $m_f : [0, +\infty) \to \mathbf{R}$ by $m_f(t) = \sup\{|f(z)| : |z| = t\}$. Show that m_f is strictly increasing on $[0, +\infty)$.

2 (1/6) Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function. Show that there exist uniquely determined entire functions f_1 and f_2 satisfying the following two conditions:

(a) $f = f_1 + if_2$ on $\mathbf{C}$ and

(b) f_1 and f_2 are real valued on $\mathbf{R}$.

3 (1/6) Let U be the open set $U = \mathbf{C} \setminus \{0, -1, -2, -3, \ldots\}$, i.e., remove zero and the negative integers from $\mathbf{C}$. Assume $f : U \to \mathbf{C}$ is holomorphic, $f(1) = 1$ and $zf(z) = f(z+1)$ for all $z \in U$. Show that f has a simple pole at each point $m \in \{0, -1, -2, -3, \ldots\}$ and that the residue of f at $m = -n$ is given by

$$\mathrm{Res}_{-n}(f) = \frac{(-1)^n}{n!}$$

4 (1/6) Let n be a positive integer. Determine the number of zeros of the function

$$g(z) = 2(z-1)^n - e^{-z}$$

inside the open disk $D(1,1)$ **and** show that all the zeros are of order 1. [Remember: If z_0 is a zero of g then it is enough to write g as $g(z) = (z - z_0)h(z)$ with $h(z_0) \neq 0$ in order to show that z_0 is of order 1.]

5 (1/6) Let $m \in \mathbf{N}$. Calculate the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x+x^2+\cdots+x^{2m}} = \int_{-\infty}^{+\infty} \frac{1-x}{1-x^{2m+1}}\,dx.$$

6 (1/6) Describe all the automorphisms of the first quadrant

$$Q_1 = \{z : \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$$

in terms of the 2x2 real matrices with determinant 1. [Remember: All the automorphisms of the upper half plane are of the form $\frac{az+b}{cz+d}$ where $a,\ b,\ c,\ d$ are real with $ac - bd = 1$.]