Math 128a - Final - Spring 2002 J. Demmel

This exam is open book, open notes, open calculator (you shouldn't need one). The total score is 130 points. The number of points approximately indicates the number of minutes you should spend on the problem.

1) (35 points) In this problem we explore how ODE solvers are designed.

Part A. (15 points) Use the method of undetermined coefficients to derive an Adams-Moulton method of the form

$$x_{n+1} = x_n + h * [A \cdot f_{n+1} + B \cdot f_n + C \cdot f_{n-1}]$$

Here the notation is that $t_n = n * h, x_n = x(t_n)$ and $f_n = f(t_n, x_n)$. Compute the values of A, B and C; show your work. What is the value of k such that the LTE $= O(h^k)$?

Part B. (10 points) Use the method of undetermined coefficients to derive an Adams-Bashforth method of the form

$$x_{n+1} = x_n + h * [D \cdot f_n + E \cdot f_{n-1}]$$

Compute the values of D and E; show your work. What is the value of k such that the LTE $= O(h^k)$?

Part C. (10 points) Describe an algorithm (with pseudocode) that uses the methods in Part A and B with fixed step size h to solve the ODE $x'(t) = f(t, x(t))$, starting at $x(0) = x_0$ up to time t_{final}. You may assume that t_{final} is an integer multiple of h. Make sure to describe how to monitor the LTE (but not to change h).

2) (35 points) In this problem we explore how to efficiently solve linear systems of equations $Ax = b$, when A is *banded*, i.e. only has nonzero entries near the diagonal. We say that A has *lower bandwidth lbw* if $a_{ij} = 0$ whenever $i > j + lbw$, and that A has *upper bandwidth ubw* if $a_{ij} = 0$ whenever $j > i + ubw$. For example, the 8-by-8 matrix below has $lbw = 2$ and $ubw = 3$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & 0 & 0 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & 0 \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} & a_{78} \\ 0 & 0 & 0 & 0 & 0 & a_{86} & a_{87} & a_{88} \end{bmatrix}$$

We call the part of the A that may be nonzero the *band* of A.

Part A. (10 points) Assume the n-by-n matrix band matrix A with lower bandwidth lbw and upper bandwidth ubw is stored in memory exactly as shown above. Give pseudocode for Gaussian elimination with no pivoting (GENP) that performs no arithmetic on the zero entries outside the band. (Your code should just compute the entries of L and U so that $A = L * U$).

Part B. (5 points) Considering L from Part A as a band matrix, what are its lower and upper bandwidths? Considering U from Part A as a band matrix, what are its lower and upper bandwidths?

Part C. (10 points) Band matrices are used when lbw and ubw are both much smaller than n, because the algorithm in Part A does much less work than plain Gaussian elimination. How many arithmetic operations does your algorithm from Part A do? Count additions, subtractions, multiplications and divisions each as 1 operation. Give your answer in the form $c_1 \cdot lbw \cdot ubw \cdot n + c_2 \cdot lbw \cdot n + c_3 \cdot ubw \cdot n + O(1)$, where you supply the constants c_1, c_2 and c_3. The $O(1)$ term is independent of n, but can depend on lbw and ubw, which we are assuming are small. Show how you determined these constants.

Part D. (5 points) Give pseudocode for solving $Ax = b$ using the L and U factors computed from Part A, doing no arithmetic on the zero entries of L and U.

Part E. (5 points) How many arithmetic operations does your algorithm from Part D do? Follow the same advice as for part C. Give your answer in the form $d \cdot lbw \cdot n + e \cdot ubw \cdot n + f \cdot n + O(1)$, where you supply the constants d, e and f. Show how you determined these constants.

3) (35 points) In this problem we investigate the accuracy of ODE solvers. Consider the implicit second order integration formula for $x'(t) = f(x(t))$: $x_{n+1} = x_n + hf(x_{n+1})$, $h > 0$, where x_n is the approximate solution of the ODE at $t = h \cdot n$. Consider applying this formula to the differential equation $x'(t) = \mu x(t)$, where μ is a constant and $x(0) \neq 0$ is given. μ may be any complex number $\mu = \mu_r + i \cdot \mu_i$, where $i = \sqrt{-1}$ and μ_r and μ_i are real.

Part A. (5 points.) Write down an explicit expression for x_n (the numerical solution from the formula) in terms of $x_0 = x(0)$, n, h and μ.

Part B. (5 points.) Write down an explicit expression for $x(t)$ (the true solution) in terms of $x(0)$, μ and t.

Part C. (5 points.) Under what conditions on μ does $\lim_{t\to\infty} |x(t)| = 0$ for any $x(0) \neq 0$?

Part D. (5 points.) Under what conditions on μ does $\lim_{t\to\infty} |x(t)| = \infty$ for any $x(0) \neq 0$?

Part E. (5 points.) Under what conditions on μ and h does $\lim_{n\to\infty} |x_n| = 0$ for any $x_0 \neq 0$? Give your answer in the form "The limit is 0 if and only if the complex number $\mu \cdot h$ lies in region C of the complex plane, where C is precisely described as follows ..."

Part F. (5 points.) Under what conditions on μ and h does $\lim_{n\to\infty} |x_n| = \infty$ for any $x_0 \neq 0$? Give your answer in the form "The limit is infinite if and only if the complex number $\mu \cdot h$ lies in region D of the complex plane, where D is precisely described as follows ..."

Part G. (5 points.) Assume $x(0) = x_0 \neq 0$. Complete the following sentence and explain why it is true: "$\lim_{t\to\infty} |x(t)| = \lim_{n\to\infty} |x_n|$ if and only if the complex number $\mu \cdot h$ lies in region E of the complex plane, where E is precisely described as follows..."

4) (25 points) In class we talked about *Least Squares Problems:* Let $\|r\|_2 = \sqrt{\sum_{i=1}^{n} r_i^2}$ be the length of the vector r. Then if A is an m-by-n matrix with $m > n$, b is an m-by-1 vector, the vector s that minimizes $\|A \cdot s - b\|_2$ is given by $s = (A^T A)^{-1} A^T b$.

We will use this fact to solve the following approximation problem: Suppose we are given m points in $\mathbf{R}^3$: $(x_1, y_1, z_1), \ldots, (x_m, y_m, z_m)$. Using this data, we want to find a simple function $f(\cdot,\cdot)$ of two variables such that $z_i \approx f(x_i, y_i)$, i.e. $f(x, y)$ is a good approximation of z in the sense that $\sqrt{\sum_{i=1}^{m} (f(x_i, y_i) - z_i)^2}$ is minimized.

Part A. (10 points) Suppose we want f to be a linear function: $f(x, y) = s_1 \cdot x + s_2 \cdot y + s_3$. For what matrix A and vector b is the solution given by

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = (A^T A)^{-1} A^T b$$

Part B. (10 points) Suppose we want f to be a quadratic function:

$$f(x, y) = s_1 \cdot x^2 + s_2 \cdot x \cdot y + s_3 \cdot y^2 + s_4 \cdot x + s_5 \cdot y + s_6$$

For what matrix A and vector b is the solution given by

$$s = \begin{bmatrix} s_1 \\ \vdots \\ s_6 \end{bmatrix} = (A^T A)^{-1} A^T b$$

Part C. (5 points) Suppose that you compute z_i with the program

for $i = 1$ to m
    $z_i = 37x_i^2 - 22x_i y_i + 18y_i + 10 + r_i$
end

where r_i is a random number in the range $[-1, 1]$. Suppose you then compute A, b and s as described in Part B. Give a guaranteed upper bound on the error $\|As - b\|_2$.