Math 128A - Final Exam

Spring 2000 – Nate Brown

1) (10pts) Assume M is a constant, N is a function of h and $M = N(h) + h^2 + h^3 + h^4 + \cdots$. Use Richardson's extrapolation to find a function $\tilde{N}$ and constants $K_3, K_4, \ldots$ such that $M = \tilde{N}(h) + K_3 h^3 + K_4 h^4 + \cdots$. (Hint: Replace h with $2h$.)

2) (10pts) Use a divided difference table to find a cubic polynomial whose graph passes through the points $(-1, 2)$, $(0, 1)$, $(1, 0)$ and $(2, 5)$.

3) (20pts) Use Taylor's theorem to prove that if $f \in C^2[a, b]$, $x_0 \in [a, b]$ and $h > 0$ is such that $x_0 + h \in [a, b]$ then there exists $\xi \in [x_0, x_0 + h]$ such that

$$f'(x_0) = 1/h\Big(f(x_0 + h) - f(x_0)\Big) - (f''(\xi)/2)h.$$

4) Let $f(x) = x^2 + (1/x)$. Let Q_n be the Newton polynomial and H_{2n+1} be the Hermite polynomial which interpolate f at distinct points $x_0, \ldots, x_n \in [2, 3]$.

a) (10pts) Find an n such that $|f(x) - Q_n(x)| < 10^{-6}$ for all $x \in [2, 3]$.

b) (10pts) Find an n such that $|f(x) - H_{2n+1}(x)| < 10^{-6}$ for all $x \in [2, 3]$.

5) (20pts) Use the first Lagrange interpolating polynomial (at the endpoints) to prove the simple Trapezoidal rule. That is, prove that if $f \in C^2[a, b]$ then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b)) + \frac{(b-a)^3}{12} f''(\xi).$$

(Use the identities $\int_a^b \frac{x-a}{b-a}dx = \int_a^b \frac{x-b}{a-b}dx = \frac{b-a}{2}$ and $\int_a^b (x-a)(x-b)dx = \frac{(b-a)^3}{6}$.)

6) Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$.

a)(10pts) Find elementary matrices A_1, A_2 such that $A_2 A_1 A$ is upper triangular.

b)(10pts) Let $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find a vector $\vec{c}$ and an *upper triangular* matrix $\tilde{A}$ such that if $\vec{x}$ is a solution to $\tilde{A}\,\vec{x} = \vec{c}$ then $\vec{x}$ is also a solution to $A\,\vec{x} = \vec{b}$.

7) a)(10pts) If $T \in M_n(\mathbb{R})$, $\vec{c} \in \mathbb{R}^n$ describe an iterative technique for solving the fixed point type problem $\vec{x} = T\,\vec{x} + \vec{c}$.

b)(10pts) Describe the Jacobi method for solving linear systems.

c)(5pts) Will the Jacobi method converge if $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$?

8)(10pts) Assume $A \in M_n(\mathbb{R})$ is invertible and we wish to solve the linear system $A\,\vec{x} = \vec{b}$. In general, the Gauss-Seidel method applied to this system will not converge. Find a linear system $B\,\vec{x} = \vec{c}$ with the properties that *i)* if $\vec{x}$ is a solution to the equation $B\,\vec{x} = \vec{c}$ then $\vec{x}$ is also a solution to the equation $A\,\vec{x} = \vec{b}$ and *ii)* the Gauss-Seidel method will converge for the equation $B\,\vec{x} = \vec{c}$.

9) Assume $y(t)$ is the unique solution to the initial value problem $\frac{dy}{dt} = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$. Let $h = (b-a)/n$, $t_i = a + ih$ $(1 \le i \le n)$, $w_0 = \alpha$ and $w_{i+1} = w_i + h\phi(t_i, w_i, h)$, $0 \le i \le n-1$, for some function ϕ.

a)(5pts) Define the local truncation error.

b)(5pts) Define consistency.

c)(5pts) Define convergence.

d)(20pts) If $f(t,y) = t^2e^t - 3t^3y$, prove that the modified Euler method will converge. (Recall that for this method $w_{i+1} = w_i + h/2[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$.)

10) a)(5pts) Let $\{y_n\}$ be a convergent sequence of numbers with limit y. Define what it means for $\{y_n\}$ to converge quadratically (i.e. with order of convergence 2).

b)(25pts) Assume $f \in C^\infty[a,b]$, $p \in [a,b]$, $f(p) = 0$ and $f'(p) \ne 0$. Prove that there exists $\delta > 0$ such that for any initial guess $x_0 \in [p-\delta, p+\delta]$, Newton's method will converge (at least) quadratically to p.