M. Rieffel | Mathematics 128A | December 18, 1997

Final Examination

SHOW YOUR WORK COMPLETELY AND NEATLY. **Total points = 140.**

15 1. Let p be the polynomial which interpolates $f(x)=\sqrt{x}$ at the points 4, 9, and 16. Find an upper bound for $|f(x)-p(x)|$ for $9 \le x \le 10$. Justify your answer.

2. Let $A=\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$.

15 a) Find the LU decomposition of A. .

9 b) Use the LU decomposition of A to solve the equation $Ax = b$ for $b=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

15 c) Obtain an estimate for the condition number of A for a norm of your choice (specify it). Justify your answer.

3.

4 a) Describe briefly the strategy for deriving the Runge-Kutta methods for solving ODE's.

14 b) Explain precisely the derivation, following your strategy above, of the second order Runge-Kutta method

$$y_{j+1} = y_j + h(f(x_j, y_j) + f(x_j + h,\ y_j + hf(x_j, y_j))/2 \ .$$

3 c) Define what is meant by the local truncation error for a single-step method for solving ODE's.

12 d) Show that the local truncation error for the above method is of order h^3.

(over)

4. a) Give a brief but precise geometric explanation of how one
8 obtains the formula for the Newton-Raphson method for finding the zeros of a function. (Include a precise statement of the formula).

b) Define precisely what it means for a convergent sequence of
3 numbers to converge quadratically.

c) Show precisely why the Newton-Raphson method often gives
12 quadratic convergence.

5. The Laguerre polynomials, L_n, are orthogonal polynomials on
15 $[0, +\infty)$ for the weight function $w(x)=e^{-x}$. They satisfy the beautiful recursion relation

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x),$$

while $L_0(x)=1$ and $L_1(x)=1-x$.

Derive the Gaussian two-point integration formula for

$$\int_0^\infty f(x)e^{-x}dx \; .$$

6. Suppose you are equipped with a pocket calculator with trig
15 functions (but without an integrate key), and you need to find $\int_0^1 \cos(x^2)\,dx$ with an error of $< 10^{-4}$. Explain precisely how you would proceed to do this, e.g. at what points you would evaluate $\cos(x^2)$ and what you would do with the values. (You do not need to carry out the computation.) Prove that your procedure will give an answer of the required accuracy.