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Spring 1994, Math 114

Second Midterm Exam

8 April, 1994
2:10-3:00 PM

1. (30 points) Let n be a positive integer, K a field in which the polynomial $t^n - 1$ splits, and α an element of a field extension of K such that $\alpha^n \in K$. Prove that $K(\alpha):K$ is normal, and that $\Gamma(K(\alpha):K)$ is abelian.

2. (20 points) Let p be a fixed prime.
(a) (10 points) If G is a finite group, define what is meant by a Sylow p-subgroup of G.

(b) (10 points) We have proved that every finite group G has a Sylow p-subgroup, and that any two Sylow p-subgroups of G are conjugate. What statements do these results yield about subextensions of a finite separable normal field extension $L:K$, on applying the Fundamental Theorem of Galois Theory? (Statements only; no proofs or arguments required.)

3. (20 points) Let G be a simple group, and d an integer >1 such that G has an element of order d. Show that G is generated by the set X of all its elements of order d.

4. (30 points) In all three parts of this problem, assume $L:K$ is a field extension, and $\alpha, \beta \in L$ are two elements each of which generates the extension: $K(\alpha) = L = K(\beta)$.
(a) (8 points) Show that if α is algebraic over K, then β is also algebraic, and of the same degree.

(b) (8 points) Show by example that α and β can be algebraic with different minimal polynomials over K.

(c) (14 points) Prove that if α is separable over K, then so is β.