MATH 110 – FINAL EXAM

PROFESSOR TARA S. HOLM

December 12, 2002

Time limit: 180 minutes (3 hours)

Name:

SID:

Please put away all books, papers, calculators, pagers, and other electronic devices. Please write your name on each page of the exam. Don't trust staples to hold the exam together.

Please write all answers in **complete sentences**. You will not receive full credit if your answers are not in complete sentences. The final answer of a computation should also be written as a complete sentence.

There are two extra blank pages at the end of the exam. You may use these for computations, but **I will not read them.** Please transfer all final answers to the page on which the question is posed.

Problem:	Your score:	Total points
1		60 points
2		30 points
3		20 points
4		18 points
5		12 points
6		20 points
7		20 points
8		20 points
Total:		200 points

1. **True/False.** Please label each of the following statements as *True* or *False*. Please explain your answer: give a brief justification or a counterexample.

   (a) **(15 points)** For the class of all vector spaces, the relation "V is isomorphic to W" is an equivalence relation.

   (b) **(15 points)** Suppose that V and W are vector spaces, and $T : V \to W$ is a linear transformation. Then T maps linearly independent subsets of V onto linearly independent subsets of W.

(**True/False.** Follow the instructions above.)

(c) **(15 points)** Let V be a finite dimensional inner product space, and let $S_1 \subseteq S_2$ be subsets of V. Then $S_1^\perp \subseteq S_2^\perp$.

(d) **(15 points)** Let $A \in M_{n\times n}(\mathbb{R})$, and let $Row(A)$ denote the subspace of $\mathbb{R}^n$ spanned by the rows of A. Then $\mathbb{R}^n = N(A) \oplus Row(A)$.

2. Let V be the set of all 2×2 matrices with entries in $\mathbb{C}$ the complex numbers, such that $A_{11} + A_{22} = 0$.
   (a) **(15 points)** Show that V is a vector space over $\mathbb{F} = \mathbb{R}$ the *real numbers*, with respect to the usual matrix addition and scalar multiplication.

   (b) **(15 points)** Find a basis for this vector space (over $\mathbb{R}$).

3. (**20 points**) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & c & 3 \end{bmatrix}.$$

For what values of c does the system of linear equations $Ax = b$ have one solution? no solutions? infinitely many solutions? Show an example of each of these possibilities, if possible. (Your example should include a choice of $b \in \mathbb{R}^3$.)

4. **(18 points)** Consider the basis of $\mathbb{R}^3$

$$\beta = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} = \{v_1, v_2, v_3\}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that fixes v_1 and v_2 and sends v_3 to $-v_3$. Compute the matrix of T with respect to the standard basis of $\mathbb{R}^3$. Can you give a geometric interpretation of T?

5. **(12 points)** Suppose that A and B are $n \times n$ matrices with the property that

$$AB = -BA.$$

Prove that if n is odd, then either A or B is *not* invertible.

6. **Eigenvalues and eigenvectors.**

(a) **(10 points)** Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -3 & 6 \\ 6 & 2 \end{bmatrix}.$$

(b) **(10 points)** For the matrices

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix},$$

find a matrix Q of eigenvectors, and matrices D_1 and D_2 of eigenvalues such that $B = QD_1Q^{-1}$ and $C = QD_2Q^{-1}$. (You are simultaneously diagonalizing B and C.)

NAME:

7. **(20 points)** Let V be a finite dimensional inner product space, $U \subseteq V$ a subspace of V, P_U orthogonal projection onto U, and $T : V \to V$ a linear operator. Prove that U and $U^\perp$ are T-invariant subspaces $\iff$ $T \circ P_U = P_U \circ T$.

8. **(20 points)** Let V be a finite dimensional vector spaces over $\mathbb{F}$. Recall that for a subset $W \subseteq V$, the annihilator of W is

$$W^0 = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}.$$

Suppose now that W is a subspace of V. Prove that

$$\dim(W) + \dim(W^0) = \dim(V).$$