MATH 170 J.Mihalisin **MIDTERM EXAMINATION** AUTUMN 2002

$\boxed{1}$ (10 points)
Do JUST ONE of the following two problems:
(Cross out the one you DON'T attempt)

(I).

Draw the Schlegel Diagram for each of the following two polytopes:

(a) a pyramid over a pentagon
(b) a prism over a hexagon

(II).

Convert the following problem into Standard Form:

minimize $\vec{c} \cdot \vec{x}$

Subject to:

$$\vec{a}_i \cdot \vec{x} \geq b_i, \quad i \in M_1$$
$$\vec{a}_i \cdot \vec{x} \leq b_i, \quad i \in M_2$$
$$\vec{a}_i \cdot \vec{x} = b_i, \quad i \in M_3$$

$$\vec{x}_j \geq \vec{0}, \quad j \in N_1$$
$$\vec{x}_j \leq \vec{0}, \quad j \in N_2$$

(The previous page is blank to give room for your answer.)

MATH 170 — MIDTERM EXAMINATION — AUTUMN 2002

$\boxed{2}$ (20 points)
Do JUST ONE of the following two problems:
(Cross out the one you DON'T attempt)

(I). Consider the polyhedron:

$$P = \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_i \cdot \vec{x} \geq b_i, \quad i = 1, 2, \ldots, m\}$$

Suppose that $\vec{u}$ and $\vec{v}$ are distinct basic feasible solutions that satisfy:

$$\vec{a}_i \cdot \vec{u} = \vec{a}_i \cdot \vec{v} = b_i \quad \text{for} \quad i = 1, 2, \ldots, (n-1)$$

and that the vectors $\{\vec{a}_i : i = 1, 2, \ldots, (n-1)\}$ are linearly independent.

Let $L = \{\lambda\vec{u} + (1-\lambda)\vec{v} \mid 0 \leq \lambda \leq 1\}$ be the segment joining the two vertices.

Prove that $L = \{\vec{z} \in P \mid \vec{a}_i \cdot \vec{z} = b_i, \quad i = 1, 2, \ldots, (n-1)\}$.

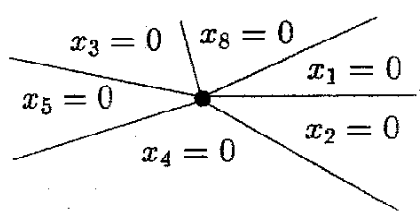
(II).

The diagram below shows a vertex of a 3-d polyhedron (described via the standard form in $\mathbb{R}^8$). Its active constraints are labelled to indicate which variable is set equal to zero.

(a) Write a basis for which the x_3 direction is NOT feasible.
(The "x_3 direction" is the direction corresponding to $\overline{c}_3$.)

(b) Write two different bases for which the x_3 direction is a feasible direction.

(c) Do your two answers to (b) give the same direction?



(The previous page is blank to give room for your answer.)

MATH 170 **MIDTERM EXAMINATION** AUTUMN 2002

$\boxed{3}$ (20 points)
Do JUST ONE of the following two problems:
(Cross out the one you DON'T attempt)

(I).

While solving a standard form problem, we arrive at the following tableau, with x_3, x_4 and x_5 being the basic variables:

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries α, β, γ, δ, η are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.

(a) The current solution is feasible but not optimal.

(b) The optimal cost is $-\infty$.

(c) The current solution is optimal and there are multiple optimal solutions.

(II).

Consider the polyhedra:

$$P = \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_i \cdot \vec{x} \leq b_i, \quad i = 1, 2, \ldots, m_P\}$$

and

$$Q = \{\vec{x} \in \mathbb{R}^n \mid \vec{c}_i \cdot \vec{x} \leq d_i, \quad i = 1, 2, \ldots, m_Q\}$$

Devise an algorithm consisting of one or more linear programs that first determines whether P and Q intersect and if they don't intersect returns the equation for a hyperplane which separates the two polyhedra. (Be SURE it produces the complete equation for the hyperplane, not just the normal vector.)

(The previous page is blank to give room for your answer.)

MATH 170 MIDTERM EXAMINATION AUTUMN 2002

$\boxed{4}$ (20 points)

Do JUST ONE of the following two problems:
(Cross out the one you DON'T attempt)

(I).

You are trying to optimize profit for a factory that produces n different products: x_1, x_2, $\ldots$, x_n. There are m different constraints (the amount of various machine times, assembly times and supervisory times available for the entire line of products). One unit of the i^{th} product requires a_{ij} worth of the j^{th} constraint. The total amount available for the j^{th} constraint is b_j. For each product, the selling price of the product minus the base cost is given by c_i.

(a) Assuming unlimited demand, formulate a linear program to maximize the profit for your factory.

For the following questions, assume that you've run the simplex algorithm to solve the program and that the first m variables form an optimal basis. Let B be the m x m matrix formed by $\{a_{ij} \mid 1 \leq i, j \leq m\}$.

(b) If the base price of the j^{th} product increases by one cent (assuming that c_j is in units of cents and that this is a relatively small change), how much will the total profit decrease?

(c) Explain qualitatively why it is necessary to assume the increase in part (b) above is "relatively small".

(d) In terms of the given quantities, determine the maximum amount you can pay per unit for a (relatively small) additional quantity of the i^{th} constraint. (I.e., how much can you pay to increase b_i to $b_i + 1$ without decreasing the total profit.)

(e) In terms of the given quantities, what is the actual profit per unit of the j^{th} product? (i.e., its selling price minus the base price as well as all the associated costs of producing it)

(II).

Let $\vec{x}_1$ be a vertex of the polyhedron P.

Consider the parametric programming problem:

$$\text{minimize } (\vec{c} + \theta\vec{d}) \cdot \vec{x} \text{ for } \vec{x} \in P.$$

If $\theta_2 > \theta_1$ and $\vec{x}_1$ is optimal for cost $(\vec{c} + \theta_1\vec{d})$ but is NOT optimal for cost $(\vec{c} + \theta_2\vec{d})$, prove that $\vec{x}_1$ is NOT optimal for the cost $(\vec{c} + \theta_3\vec{d})$ for ANY $\theta_3 > \theta_2$.

(The previous page is blank to give room for your answer.)