## MATH 121B: FINAL, SPRING, 2000

There are six problems on this exam. Do problems 1-4, and **one of problems 5-6**. If you choose to do both problems 5 and 6, your final score will be the sum of your total score on problems 1-4, and the higher one of your scores on Problems 5 and 6.

Total score: 175 points.

**Problem 1.** *In spherical coordinates arclength is given by*

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

- (i) *(10 points) Set up the Euler-Lagrange equation for a geodesic on the cone $\theta = \alpha$, $\alpha$ a constant.*
- (ii) *(10 points) Find the first integral of these equations, and write it in the form $d\phi/dr = \dots$.*
- (iii) *(7 points) Show that the functions $\phi(r) = \phi_0$, $\phi_0$ a constant, solve this equation. What curves on the cone are the corresponding geodesics?*
- (iv) *(8 points) Solve the first order ODE $d\phi/dr = \dots$ that you obtained in (ii). (Hint: change variables in the integral.)*

**Problem 2.** *Consider the ODE*

$$x^2y'' + 2xy' + (x^2 - 2)y = 0.$$

- (i) *(20 points) Using Frobenius' method of generalized (fractional) power series (around $x = 0$), find two linearly independent solutions of the ODE. State explicitly the leading power of the power series, as well as the recursion relations for the coefficients.*
- (ii) *(15 points) Let $y = x^{-1/2}u$. Substituting $y$ into the ODE, obtain an ODE for $u$. What are its solutions? (Hint: look at the Bessel ODE!)*

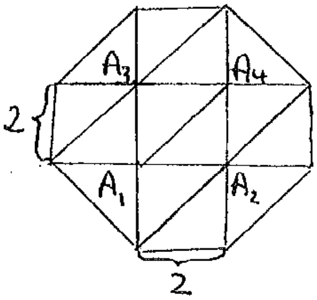
**Problem 3.** *We wish to find the steady state temperature $u$ inside a sphere of radius $a$ whose surface is kept at a given temperature $u_0 = u_0(\theta, \phi)$. (The Laplacian in spherical coordinates is given among the formulae.)*

- (i) *(15 points) Separating $r$ and the spherical variables, show that the general solution of $\Delta u = 0$ inside the sphere is given by*

$$u = \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} r^l P_l^m(\cos\theta)(A_{ml}\cos(m\phi) + B_{ml}\sin(m\phi)).$$

- (ii) *(10 points) Suppose $u_0$ is independent of $\phi$. Find the constants $A_{ml}$, $B_{ml}$ in terms of $u_0$. Your answer should read $A_{ml} = \dots$ where $A_{ml}$ and $B_{ml}$ do not appear on the right hand side.*
- (iii) *(10 points) Now suppose $u_0(\theta, \phi) = \cos^2\theta$. Find $u$ explicitly.*

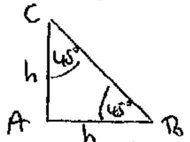
**Problem 4.** *Using the method of finite elements, solve (approximately) the equation $-\Delta u = f$ in a disk $D$ of radius 3, with $u = 0$ on the boundary of $D$, and $f \equiv 1$ on $D$. More specifically, divide up an approximation of the region $D$ into small triangles (elements) as shown, with the interior vertices labelled as $A_j$, $j = 1, \ldots 4$.*



*Recall that on an element with vertices $A$, $B$ and $C$, as shown below, the element matrix $k_e$ is*

$$k_e = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} A & B & C & \\ \end{array}$$

$$k_e = \begin{array}{cccc} A & B & C & \\ 1 & -1/2 & -1/2 & A \\ -1/2 & 1/2 & 0 & B \\ -1/2 & 0 & 1/2 & C \end{array}$$



(i) *(7 points) Show that the weak form of the PDE is $Q(u, v) = 0$ for all $v$ satisfying the boundary condition, where*

$$Q(u,v) = \int\int_D \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - vf\right] dx\, dy.$$

(ii) *(10 points) For $U$ in our space of trial functions (i.e. $U(x,y) = \sum_{i=1}^{4} U_i T_i(x,y)$, where $T_i$ is linear on each element, continuous on $D$, vanishes on the boundary of $D$ and at every interior vertex except $A_i$, where it is 1), $v = T_j$, $j = 1, \ldots, 4$, the above expression gives the matrix equation $KU = F$. Here $U^T = [U_1, \ldots U_4]$, $U_j = U(A_j)$.*

*Using the $k_e$ given above, or otherwise, show that the matrix $K$ is*

$$K = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

(iii) *(8 points) Approximating the integral over each element by the area of the element times the value of the integrand at the centroid, find $F$. (Recall that a linear function on a triangle, evaluated at the centroid of the triangle, gives the average of its values at the three vertices.)*

(iv) *(10 points) Solve $KU = F$ to find the value of $U(x,y)$ at the vertices. (Hint: you can simplify your task by noticing that some $U_j$ will be equal to each other.) What is $U$ at the origin?*

**Problem 5.** *We wish to solve $\Delta u = -f$ on a disk $D$ of radius $a$ with $u = 0$ on the boundary of the disk, where $f$ is a given function. (This problem represents, for example, finding the steady state temperature in a metal plate that has a heat source in it. represented by $f$; the Laplacian in polar coordinates is given among the formulae.) Proceed as follows.*

- (i) *(15 points) First, using separation of variables, find the eigenfunctions and eigenvalues of $\Delta$ on $D$ with homogeneous Dirichlet boundary conditions, i.e. find $\lambda$ and $u$ such that $\Delta u = -\lambda u$, and $u = 0$ when $r = a$.*
- (ii) *(5 points) If $u_{nm}$ are the eigenfunctions with $\Delta u_{nm} = -\lambda_{nm} u_{nm}$, find $\Delta u$ for $u = \sum_{nm} c_{nm} u_{nm}$.*
- (iii) *(8 points) Substitute $u = \sum_{nm} c_{nm} u_{nm}$ into the PDE, $-\Delta u = f$, and obtain an equation expressing $c_{nm}$ in terms of $f$. Your result should state $c_{nm} = \ldots$, where the constants $c_{nm}$ do not appear on the right hand side.*
- (iv) *(7 points) If $f \equiv 1$ on $D$, find $c_{nm}$ explicitly in terms of values of the Bessel functions (i.e. expressions such as $J_p(k)$ can appear in your answer.).*

**Problem 6.** *We wish to solve the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ on the real line, $\mathbb{R}_x$, with given initial temperature $u(x, 0) = \phi(x)$, $\phi$ given. Proceed as follows.*

- (i) *(7 points) Show directly from the definition of the Fourier transform (see the formulae sheet for the normalization) that if $f$ is a differentiable function which decays sufficiently at inifinity (and the same holds for its derivative) then*

$$(\mathcal{F}f')(\xi) = i\xi(\mathcal{F}f)(\xi).$$

- (ii) *(8 points) Suppose that $u$ solves the heat equation given above. Fourier transform $u$ in the $x$ variable, and show that the heat equation becomes an ODE. What is the initial condition for the ODE?*
- (iii) *(5 points) Show that the solution of the ODE is*

$$\hat{u}(\xi, t) = e^{-t\xi^2}\hat{\phi}(\xi);$$

*here $\hat{u}$ is the Fourier transform of $u$ in $x$, etc.*

- (iv) *(7 points) Take the inverse Fourier transform of this result, and obtain $u$ in terms of $\phi$.*
- (v) *(8 points) If $\phi(x) = e^{-x^2/4}$, find $u$ explicitly.*

# USEFUL FORMULAE FOR MATH 121B FINAL, SPRING, 2000

The Laplacian in polar coordinates is $(r,\theta)$ is

$$\Delta_{\mathbb{R}^2}u=\frac{\partial^2 u}{\partial r^2}+r^{-1}\frac{\partial u}{\partial r}+r^{-2}\frac{\partial^2 u}{\partial\theta^2}.$$

The Laplacian in spherical coordinates $(r,\theta,\phi)$, $0<\theta<\pi$, $0\leq\phi\leq 2\pi$, is

$$\Delta_{\mathbb{R}^3}=\frac{\partial^2 u}{\partial r^2}+2r^{-1}\frac{\partial u}{\partial r}+r^{-2}\Delta_{\mathbb{S}^2},$$

where $\Delta_{\mathbb{S}^2}$ is the Laplacian on the sphere, i.e. it does not involve $r$ or $\frac{\partial}{\partial r}$. The 'separated' eigenfunctions of $\Delta_{\mathbb{S}^2}$ are (linear combinations of) $u_{ml}=P_l^m(\cos\theta)\cos(m\phi)$, $v_{ml}=P_l^m(\cos\theta)\sin(m\phi)$, with eigenvalue $\Delta_{\mathbb{S}^2}u_{ml}=-l(l+1)u_{ml}$, and similarly for $v_{ml}$. The first few Legendre polynomials are

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x).$$

The Legendre polynomials satisfy

$$\int_{-1}^{1}P_l(x)^2\,dx=\frac{2}{2l+1}.$$

The Bessel functions $y=J_p(x)$ solve the ODE

$$x^2y''+xy'+(x^2-p^2)y=0,$$

and they satisfy $\frac{d}{dx}[x^pJ_p(x)]=x^pJ_{p-1}(x)$. In addition,

$$\int_0^a J_p(kr/a)^2r\,dr=a^2J_{p+1}(k)^2/2$$

where $k>0$ is a zero of $J_p$. The Fourier transform on $\mathbb{R}$ is

$$(\mathcal{F}f)(\xi)=\hat{f}(\xi)=\int_{-\infty}^{\infty}e^{-ix\xi}f(x)\,dx,$$

and then the inverse Fourier transform is

$$(\mathcal{F}^{-1}g)(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}g(\xi)\,d\xi.$$

It satisfies

$$\hat{f}(\xi)\hat{g}(\xi)=(\mathcal{F}(f*g))(\xi),\quad (f*g)(x)=\int_{-\infty}^{\infty}f(x-y)g(y)\,dy.$$

In addition, if $f(x)=e^{-c^2x^2}$, $c>0$, then

$$\hat{f}(\xi)=\int_{-\infty}^{\infty}e^{-ix\xi}e^{-c^2x^2}\,dx=\frac{\sqrt{\pi}}{c}e^{-\xi^2/(4c^2)}.$$

In particular, taking $\xi=0$,

$$\int_{-\infty}^{\infty}e^{-c^2x^2}\,dx=\frac{\sqrt{\pi}}{c}.$$