Math 1B Midterm
Tuesday, 22 July 2008

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Name: ____

Problem Number	1	2	3	4	5	6	7	Total
Score								
Maximum	10	10	10	15	10	20	25	100

Please do not begin this test until 8:10 a.m. The test ends exactly at 10 a.m.
As always, show work for partial credit. Please box your final answers.

1. (10 pts) Evaluate the following indefinite integral:

$$\int \frac{4x^2\,dx}{(x+1)(x-1)^2}$$

$$\begin{aligned}
\frac{4x^2}{(x+1)(x-1)^2} &= \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\
4x^2 &= A(x-1)^2 + B(x+1)(x-1) + C(x+1) \\
&= (A+B)x^2 + (-2A+C)x + (A-B+C) \\
A &= 1 \\
B &= 3 \\
C &= 2 \\
\int \frac{4x^2\,dx}{(x+1)(x-1)^2} &= \int \left[\frac{1}{x+1} + \frac{3}{x-1} + \frac{2}{(x-1)^2}\right] dx \\
&= \boxed{\ln(x+1) + 3\ln(x-1) - \frac{2}{x-1} + \mathit{constant}}
\end{aligned}$$

2. (10 pts) Evaluate the following definite integral:

$$\int_0^1 x^2 \cosh x \, dx$$

Hint: $\cosh x = (e^x + e^{-x})/2$ has easy integrals and derivatives: $\cosh' = \sinh = (e^x - e^{-x})/2$ and $\sinh' = \cosh$.

$$\begin{aligned}
\int_0^1 x^2 \cosh x \, dx &= \left[x^2 \sinh x\right]_0^1 - \int_0^1 2x \sinh x \, dx \\
&= \sinh 1 - [2x \cosh x]_0^1 + \int_0^1 2 \cosh x \\
&= \sinh 1 - 2\cosh 1 + [2 \sinh x]_0^1 \\
&= \sinh 1 - 2\cosh 1 + 2 \sinh 1 \\
&= 3\left(\frac{e^1 - e^{-1}}{2}\right) - 2\left(\frac{e^1 + e^{-1}}{2}\right) \\
&= \boxed{\frac{e}{2} - \frac{5}{2e}}
\end{aligned}$$

3. (10 pts) Compute the surface area of the surface formed by rotating the curve $\{y = x^2 : 0 \leq x \leq 1\}$ around the y-axis.

$$\begin{aligned}
\textit{Surface area around y-axis} &= \int_0^1 2\pi x\sqrt{1+(y')^2}\,dx \\
&= \int_0^1 2\pi x\sqrt{1+(2x)^2}\,dx \\
&= \int_{x=0}^1 2\pi\sqrt{1+4x^2}\,x\,dx \\
&= \int_{u=1}^5 2\pi\sqrt{u}\,\frac{du}{8} \\
&= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big]_1^5 \\
&= \boxed{\frac{\pi}{6}\left(5^{3/2}-1\right)}
\end{aligned}$$

4. (15 pts — 3 pts each) For each of the following five improper integrals, decide whether the integral converges or diverges, and box the corresponding answer. You do not need to provide any work or justification — full marks will be given for simply the correct answer.

   (a) $\displaystyle\int_{10}^{\infty} \frac{dx}{\sqrt{x}}$ CONVERGE or $\boxed{\text{DIVERGE}}$?
   possible divergence at $x = \infty$*; p-test with* $p = 1/2$

   (b) $\displaystyle\int_{0}^{1} \frac{dx}{1-x}$ CONVERGE or $\boxed{\text{DIVERGE}}$?
   possible divergence at $x = 1$*; p-test with* $p = 1$

   (c) $\displaystyle\int_{-\infty}^{0} \frac{e^x\,dx}{\arctan x}$ CONVERGE or $\boxed{\text{DIVERGE}}$?
   possible divergences at $x = -\infty$ *and* $x = 0$*; at* $x = -\infty$*, converge with* $\int_0^\infty e^{-x^2}\,dx$*; at* $x = 0$*, diverge with* $1/x$*, as* $\arctan x \approx x$ *near* $x \to 0$

   (d) $\displaystyle\int_{-\infty}^{\infty} x^3\,e^{-x^2}\,dx$ $\boxed{\text{CONVERGE}}$ or DIVERGE?
   possible divergences at $x = \pm\infty$*; u-sub* $u = x^2$*, so* $\int_0^\infty x^3 e^{-x^2} dx = \int_0^\infty u e^{-u} du/2$ *which converges*

   (e) $\displaystyle\int_{0}^{5} \frac{dx}{(x+1)\sqrt{5-x}}$ $\boxed{\text{CONVERGE}}$ or DIVERGE?
   possible divergence at $x = 5$*; p-test with* $p = 1/2$

5. (10 pts — 2 pts each) Match the following first-order differential equations to the corresponding direction fields; write the letters a–e next to the corresponding graphs. Decide (and mark by boxing) whether each differential equation is linear and/or separable.

(a) $\dfrac{dy}{dx} = \tan x \tan y$

LINEAR? $\boxed{\text{SEPARABLE}}$?

(b) $\dfrac{dy}{dx} = x^2 + 2xy$

$\boxed{\text{LINEAR}}$? SEPARABLE?

(c) $\dfrac{dy}{dx} = \sin^2 x + \sin^2 y$

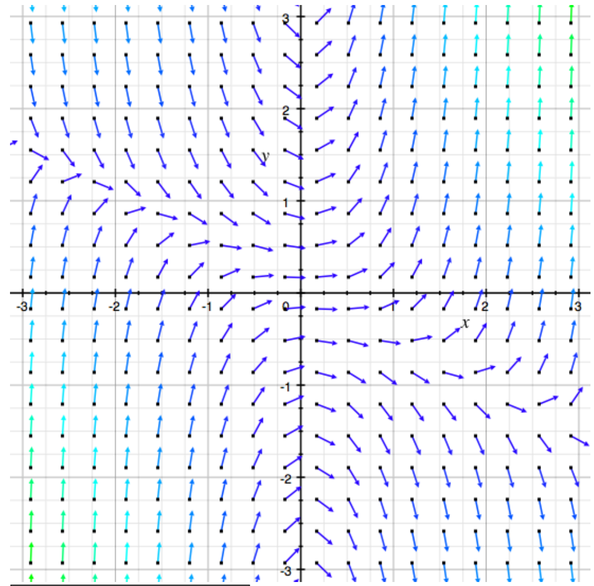
LINEAR? SEPARABLE?

(d) $\dfrac{dy}{dx} = x^2 + x^2 y$

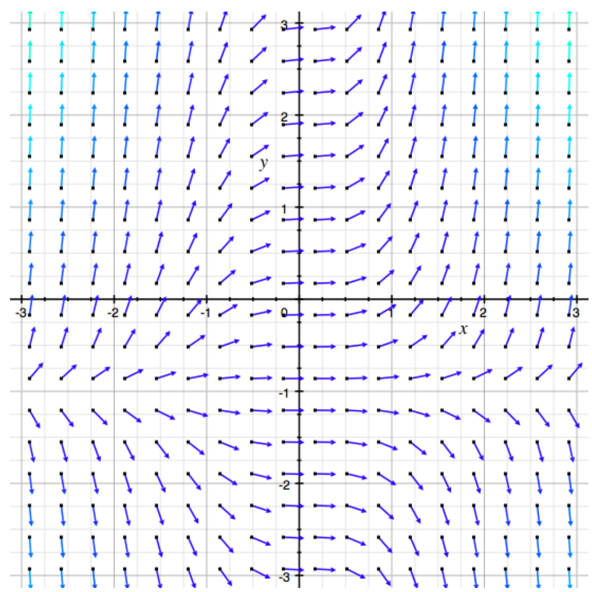
$\boxed{\text{LINEAR}}$? $\boxed{\text{SEPARABLE}}$?

(e) $\dfrac{dy}{dx} = xe^y + y$

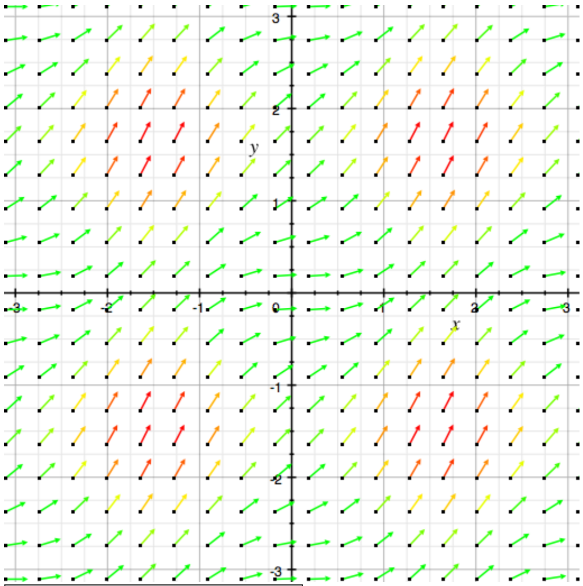
LINEAR? SEPARABLE?



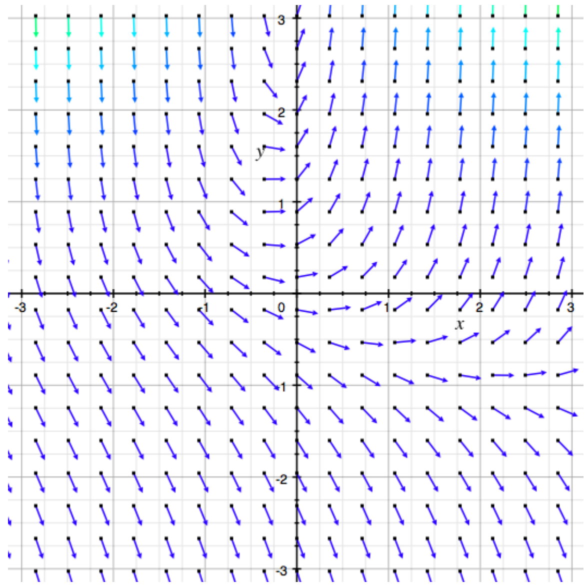
$\boxed{\text{(b) } y' = x^2 + 2xy}$



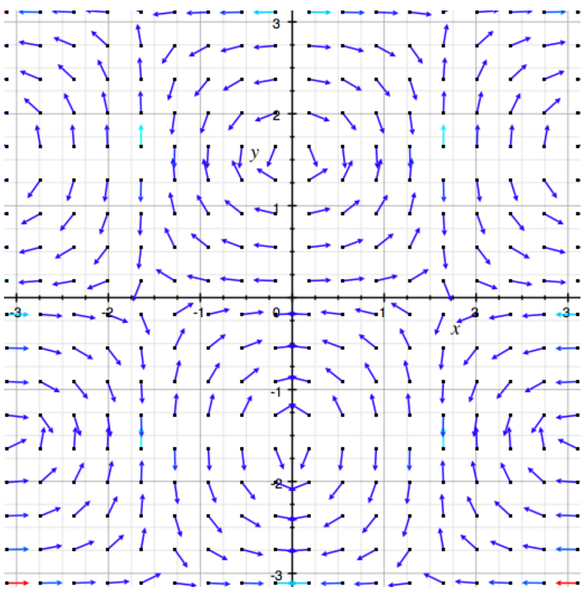
$\boxed{\text{(d) } y' = x^2 + x^2 y}$



(c) $y' = \sin^2 x + \sin^2 y$



(e) $y' = xe^y + y$



(a) $y' = \tan x \tan y$

6. (20 pts) If left to their own devices, the weeds in my garden grow at a rate that varies with the month: if $W(t)$ is the number of weeds at time t, then the growth rate is

$$W'_{\text{grow}}(t) = W(t) \times (1 + \cos(2\pi t/12 \text{ months}))\,/\text{month}.$$

But I don't leave the weeds alone: I also pull them out at a rate of

$$W'_{\text{pull}}(t) = (1 + \cos(2\pi t/12 \text{ months})) \times 20 \text{ plants/month}.$$

If my garden starts with 30 weeds in it, find the number of weeds in the garden after t months.

We can solve this as a separable equation:

$$\begin{aligned}
\frac{dW}{dt} &= W'_{\text{grow}}(t) - W'_{\text{pull}}(t) \\
&= W(t) \times (1 + \cos(2\pi t/12 \text{ months}))\,/\text{month} \\
&\quad - (1 + \cos(2\pi t/12 \text{ months})) \times 20 \text{ plants/month} \\
&= (W(t) - 20 \text{ plants}) \times (1 + \cos(2\pi t/12 \text{ months}))\,/\text{month} \\
\int \frac{dW}{W(t) - 20 \text{ plants}} &= \int (1 + \cos(2\pi t/12 \text{ months}))\,\frac{dt}{\text{month}} \\
\ln(W(t) - 20 \text{ plants}) &= \frac{1}{\text{month}}\left(t + \frac{12 \text{ months}}{2\pi}\sin(2\pi t/12 \text{ months})\right) + \textit{constant} \\
&= \left(\frac{t}{\text{month}} + \frac{6}{\pi}\sin(2\pi t/12 \text{ months})\right) + \textit{constant} \\
W(t) &= 20 \text{ plants} + \textit{constant} \times e^{\left(\frac{t}{\text{month}} + \frac{6}{\pi}\sin(2\pi t/12 \text{ months})\right)} \\
30 \text{ plants} = W(0) &= 20 \text{ plants} + \textit{constant} \times e^{\left(\frac{0}{\text{month}} + \frac{6}{\pi}\sin(2\pi 0/12 \text{ months})\right)} \\
&= 20 \text{ plants} + \textit{constant} \times 1 \\
\textit{constant} &= 10 \\
W(t) &= \boxed{20 \text{ plants} + 10 \text{ plants} \times e^{\left(\frac{t}{\text{month}} + \frac{6}{\pi}\sin(2\pi t/12 \text{ months})\right)}}
\end{aligned}$$

Alternately, we can solve this as a linear equation:

$$\begin{aligned}
\frac{dW}{dt} &= W(t) \times (1 + \cos(2\pi t/12 \text{ months}))\,/\text{month} \\
&\quad - (1 + \cos(2\pi t/12 \text{ months})) \times 20 \text{ plants/month} \\
I(t) &= e^{-\int (1+\cos(2\pi t/12 \text{ months}))dt/\text{month}} \\
&= e^{-t/\text{month} - (6/\pi)\sin(2\pi t/12 \text{ months})} \\
(IW)' &= I'W + IW' \\
&= -I(t)\,(1 + \cos(2\pi t/12 \text{ months}))\,W(t) + I(t)\,W'(t) \\
&= -I(t)\;(1 + \cos(2\pi t/12 \text{ months})) \times 20 \text{ plants/month} \\
I(t)\,W(t) &= -\int e^{-t/\text{month} - (6/\pi)\sin(2\pi t/12 \text{ months})}\,(1 + \cos(2\pi t/12 \text{ months})) \times 20\,dt \text{ plants/month} \\
&= 20 \text{ plants} \times \int e^u\,du \\
&= 20 \text{ plants} \times e^u + \textit{constant} = 20 \text{ plants} \times I(t) + \textit{constant} \\
W(t) &= \boxed{20 \text{ plants} + \textit{constant} \times e^{t/\text{month} + (6/\pi)\sin(2\pi t/12 \text{ months})}}
\end{aligned}$$

7. (25 pts) A spring with a 1-gram bob ($m = 1$) is placed in a viscous medium with friction-coefficient $c = 2$ grams per second. The spring constant is $k = 2$ grams per second squared. The spring is released from its rest position with zero velocity, but it is driven under a force $F(t \text{ seconds}) = e^{-t}/(1+\cos^2 t)$ dynes. What is the position (in centimeters) of the bob at time t? (A dyne is a "gram centimeter per second squared" — it's the unit of force in "cgs". I.e., the units in the problem are "magic" — they work out right — so you can ignore units if you wish. Of course, keeping units around is a good way to check your work.)

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\begin{aligned}
1\mathrm{g}\, y'' + 2\frac{\mathrm{g}}{\mathrm{s}}\, y' + 2\frac{\mathrm{g}}{\mathrm{s}^2}\, y &= \frac{e^{-t/\mathrm{s}}}{1+\cos^2 t/\mathrm{s}}\,\frac{\mathrm{g\,cm}}{\mathrm{s}^2} = f(t)\,\mathrm{g} \\
1\mathrm{g}\, r^2 + 2\frac{\mathrm{g}}{\mathrm{s}}\, r + 2\frac{\mathrm{g}}{\mathrm{s}^2} &= 0 \\
r &= (-1 \pm i)\,/\mathrm{s} \\
y_1(t) &= e^{-t/\mathrm{s}} \cos t/\mathrm{s} \\
y_2(t) &= e^{-t/\mathrm{s}} \sin t/\mathrm{s} \\
u_1'(t) &= \frac{f(t)\, y_2(t)}{y_1'(t)\, y_2(t) - y_2'(t)\, y_1(t)} \\
u_2'(t) &= \frac{-f(t)\, y_1(t)}{y_1'(t)\, y_2(t) - y_2'(t)\, y_1(t)} \\
y_1'(t)\, y_2(t) - y_2'(t)\, y_1(t) &= \left(-\frac{1}{\mathrm{s}} e^{-t/\mathrm{s}} \cos t/\mathrm{s} - \frac{1}{\mathrm{s}} e^{-t/\mathrm{s}} \sin t/\mathrm{s}\right) e^{-t/\mathrm{s}} \sin t/\mathrm{s} \\
&\quad - \left(-\frac{1}{\mathrm{s}} e^{-t/\mathrm{s}} \sin t/\mathrm{s} + \frac{1}{\mathrm{s}} e^{-t/\mathrm{s}} \cos t/\mathrm{s}\right) e^{-t/\mathrm{s}} \sin t/\mathrm{s} \\
&= -\frac{1}{\mathrm{s}} e^{-2t/\mathrm{s}} \left(\cos t/\mathrm{s} \sin t/\mathrm{s} + \sin^2 t/\mathrm{s} - \sin t/\mathrm{s} \cos t/\mathrm{s} + \cos^2 t/\mathrm{s}\right) \\
&= -\frac{1}{\mathrm{s}} e^{-2t/\mathrm{s}} \\
u_1'(t) &= \frac{1}{-\frac{1}{\mathrm{s}} e^{-2t/\mathrm{s}}} \frac{e^{-t/\mathrm{s}}}{1+\cos^2 t/\mathrm{s}} \frac{\mathrm{cm}}{\mathrm{s}^2} e^{-t/\mathrm{s}} \sin t/\mathrm{s} \\
&= \frac{-\sin t/\mathrm{s}}{1+\cos^2 t/\mathrm{s}} \frac{\mathrm{cm}}{\mathrm{s}} \\
u_1(t) &= \int \frac{-\sin t/\mathrm{s}}{1+\cos^2 t/\mathrm{s}} \frac{\mathrm{cm}}{\mathrm{s}}\, dt \\
&= \int \frac{du\, \mathrm{cm}}{1+u^2} \\
&= \arctan\left(\cos t/\mathrm{s}\right) \mathrm{cm} + c_1 \\
u_2(t) &= \int \frac{\cos t/\mathrm{s}}{1+\cos^2 t/\mathrm{s}} \frac{\mathrm{cm}}{\mathrm{s}}\, dt \\
&= \int \frac{\cos t/\mathrm{s}}{2-\sin^2 t/\mathrm{s}} \frac{\mathrm{cm}}{\mathrm{s}}\, dt \\
&= \int \frac{du\, \mathrm{cm}}{2-u^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \int \left(\frac{\sqrt{2}/4}{\sqrt{2}+u} + \frac{\sqrt{2}/4}{\sqrt{2}-u} \right) du \,\mathrm{cm} \\
&= \frac{\sqrt{2}}{4} \left(\ln(\sqrt{2}+u) - \ln(\sqrt{2}-u) \right) \mathrm{cm} + c_2 \\
&= \frac{\sqrt{2}}{4} \ln \left(\frac{\sqrt{2}+\sin t/\mathrm{s}}{\sqrt{2}-\sin t/\mathrm{s}} \right) \mathrm{cm} + c_2 \\
y(t) &= y_1(t)\, u_1(t) + y_2(t)\, u_2(t) \\
&= e^{-t/\mathrm{s}} \cos t/\mathrm{s} \left(\arctan\left(\cos t/\mathrm{s}\right) \mathrm{cm} + c_1\right) \\
&\quad + e^{-t/\mathrm{s}} \sin t/\mathrm{s} \left(\frac{\sqrt{2}}{4} \ln \left(\frac{\sqrt{2}+\sin t/\mathrm{s}}{\sqrt{2}-\sin t/\mathrm{s}} \right) \mathrm{cm} + c_2 \right) \\
0 = y(0) &= 1 \times 1 \times (\pi/4 + c_1) + 1 \times 0 \times (0 + c_2) \\
c_1 &= -\frac{\pi}{4} \mathrm{cm} \\
y'(t) &= -\frac{1}{\mathrm{s}} y(t) + e^{-t/\mathrm{s}} \left(-\frac{1}{\mathrm{s}} \sin t/\mathrm{s} \left(\arctan\left(\cos t/\mathrm{s}\right) \mathrm{cm} - \frac{\pi}{4}\mathrm{cm} \right) + \right. \\
&\qquad + \cos t/\mathrm{s} \frac{-\frac{1}{\mathrm{s}} \sin t/\mathrm{s}}{1 + \cos^2 t/\mathrm{s}} + \frac{1}{\mathrm{s}} \cos t/\mathrm{s} \left(\frac{\sqrt{2}}{4} \ln \left(\frac{\sqrt{2}+\sin t/\mathrm{s}}{\sqrt{2}-\sin t/\mathrm{s}} \right) \mathrm{cm} + c_2 \right) \\
&\qquad \left. + \sin t/\mathrm{s} \times \frac{d}{dt} \left[\frac{\sqrt{2}}{4} \ln \left(\frac{\sqrt{2}+\sin t/\mathrm{s}}{\sqrt{2}-\sin t/\mathrm{s}} \right) \mathrm{cm} + c_2 \right] \right) \\
0 = y'(0) &= 0 + 1 \times (0 + 0 + 1 \times (0 + c_2) + 0) \\
c_1 &= 0 \\
y(t) &= \boxed{ e^{-t/\mathrm{s}} \cos t/\mathrm{s} \left(\arctan\left(\cos t/\mathrm{s}\right) \mathrm{cm} - \frac{\pi}{4}\mathrm{cm} \right) + e^{-t/\mathrm{s}} \sin t/\mathrm{s} \frac{\sqrt{2}}{4} \ln \left(\frac{\sqrt{2}+\sin t/\mathrm{s}}{\sqrt{2}-\sin t/\mathrm{s}} \mathrm{cm} \right) }
\end{aligned}
$$

8. (0 pts) Invent a new kitchen tool/appliance, using a cone, a tube, a sphere, and a disk/wheel. Draw the tool/appliance, and explain what it does. Best new appliance gets to pick the kind of cookies for the final.

   Ask Theo for cookie recipes, if you like to bake.