CS 70 Spring 2008 — Discrete Mathematics for CS — David Wagner — MT 2

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your Unix account login: ____

Your section time (e.g., Tue 3pm): ____

Name of the person sitting to your left: ___________________________

Name of the person sitting to your right: ___________________________

You may consult any books, notes, or other paper-based inanimate objects available to you. Calculators and computers are not permitted. Please write your answers in the spaces provided in the test; in particular, we will not grade anything on the back of an exam page unless we are clearly told on the front of the page to look there.

You have 50 minutes. There are 3 questions, of varying credit (40 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Do not turn this page until your instructor tells you to do so.

Problem 1	
Problem 2	
Problem 3	
Total	

Problem 1. [True or false] (11 points)

Circle TRUE or FALSE. Do not justify your answers on this problem.

(a) TRUE or FALSE: Let $P(x)$ be any polynomial of degree 10 with integer coefficients. Then $P(x)$ has at most 10 roots modulo 19.

(b) TRUE or FALSE: Let $P(x)$ be any polynomial of degree 10 with integer coefficients. Then $P(x)$ has at most 10 roots modulo 20.

(c) TRUE or FALSE: There is a unique polynomial $P(x)$ modulo 5 of degree at most 1 such that $P(1) \equiv 2008 \pmod 5$ and $P(2008) \equiv 1 \pmod 5$.

(d) TRUE or FALSE: There is a unique polynomial $P(x)$ modulo 5 of degree at most 1 such that $P(3) \equiv 123 \pmod 5$ and $P(123) \equiv 3 \pmod 5$.

(e) TRUE or FALSE: Let $f(n)$ denote the maximum number of edges that an undirected graph with n vertices can have. Then $f(n) \in \mathrm{O}(n^2)$.

(f) TRUE or FALSE: There are exactly $\binom{10}{3}$ 10-bit strings that contain at most three 1-bits.

(g) TRUE or FALSE: There are exactly $\frac{52\times 51\times 50}{3\times 2}$ ways to select three cards from a standard deck of 52 cards, if the order in which we select those three cards doesn't matter.

(h) TRUE or FALSE: There are exactly $\frac{52\times 51\times 50}{3\times 2}$ ways to select three cards from a standard deck of 52 cards, if the order in which we select those three cards matters.

(i) TRUE or FALSE: If the events A and B are independent, then it is guaranteed that $\Pr[A\cup B] = \Pr[A] + \Pr[B]$.

(j) TRUE or FALSE: If $\Pr[A] \neq 0$ and $\Pr[B] \neq 0$ and $\Pr[A|B] = \Pr[A]$, then it is guaranteed that $\Pr[B|A] = \Pr[B]$.

(k) TRUE or FALSE: If $\Pr[A] \neq 0$ and $\Pr[B] \neq 0$, then it is guaranteed that $\Pr[B|A] = \frac{\Pr[B]}{\Pr[A]} \times \Pr[A|B]$.

Problem 2. [Proofs and polynomials] (11 points)

Define the sequence $P_1(x), P_2(x), \ldots$ of polynomials as follows: $P_1(x) = x + 12$, $P_2(x) = x^2 - 6x + 5$, and

$$P_{n+1}(x) = xP_n(x) - P_{n-1}(x) \qquad \text{for all } n \geq 2.$$

(a) Fill in the blank: $P_1(5) =$ ______

(b) Fill in the blank: $P_2(5) =$ ______

(c) Fill in the blank: $P_3(5) =$ ______

(d) Prove that $P_n(5) \equiv 0 \pmod{17}$ for every integer $n \geq 1$.

(e) Let q be a prime number. Prove that the polynomial $P_{2008}(x)$ has at most 2008 different roots modulo q.

Problem 3. [Probability] (18 points)

The card game Euchre is played with a deck of 24 cards consisting of only the 9, 10, J, Q, K, and A of each suit (i.e., we start with an ordinary 52-card deck and remove the cards 2–8). The J, Q, K, and A are considered face cards. There are four players, Debbie, Eve, Frank, and George. In this problem, show your work. You can leave your answer as an unevaluated expression.

(a) Debbie, the dealer, shuffles the deck randomly, so that all orderings are equally likely. How many ways are there to order the deck of 24 cards?

(b) After shuffling the deck, Debbie deals five cards to each player. Finally Eve turns up the top card remaining and puts it in the middle of the table. What is the probability that the card Eve turned up is an ace (A)?

(c) What is the probability that Frank is dealt a hand with no face cards?

(d) What is the probability that Frank was dealt a hand with no face cards *and* that the card Eve turned up in part (b) was an ace?

(e) What is the (conditional) probability that Frank was dealt a hand with no face cards, given that the card Eve turned up was an ace?