Name Solution

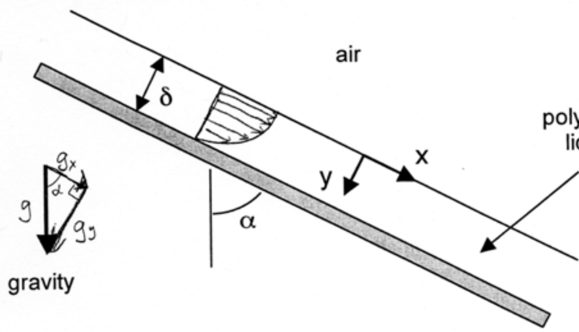
1. (45 pts) A polymeric liquid is coated onto a surface by flowing the liquid down an inclined plate making an angle α with the vertical. Assume that the polymeric liquid behaves as a power-law fluid with a viscosity given by:

$$\eta = K\left|\frac{1}{2}II\right|^{(n-1)/2}$$

where K and n are constants.

a) Obtain an expression for the **velocity distribution in the film** and for the **thickness of the film** given that the flow rate per unit width is q. ***Use the coordinate system indicated in the sketch. Indicate your assumptions and show your work.***

b) Show what your velocity field and film thickness expressions reduce to in the limit that the fluid behaves as a Newtonian fluid. **Do not re-solve** the entire problem for a Newtonian fluid.



$$\boxed{\begin{aligned} g_x &= g\cos\alpha \\ g_y &= g\sin\alpha \end{aligned}}$$

a)

Assume: steady-state
incompressible fluid
Fully-developed flow so $\underline{v} \neq \underline{v}(x)$
all flow is parallel to wall so $v_y = v_z = 0$
$\delta \ll W$ so $v_x = v_x(y)$ only
assume air is inviscid, i.e. $\eta_{air} \approx 0$

Check continuity: $\cancel{\frac{\partial v_x}{\partial x}} + \frac{\partial v_y}{\partial y}^{\nearrow 0} + \frac{\partial v_z}{\partial z}^{\nearrow 0} = 0 \Rightarrow 0 = 0$ ✓ Consistent w/ continuity.

since $v_x = v_x(y)$ only

Power-law fluid so: $\tau_{xx} = \eta(II)\left[2\frac{\partial v_x}{\partial x}\right]$ etc.

looking at table 7.3 & from $v_x = v_x(y)$ only, see that only non-zero stresses are

$$\tau_{yx} = \tau_{xy} = \eta(II)\frac{dv_x}{dy}$$

Name Solution

From Table 8-1 $\quad \frac{1}{2}\mathrm{II} = \left(\frac{dv_x}{dy}\right)^2 \Rightarrow \eta(\mathrm{II}) = K\left|\frac{dv_x}{dy}\right|^{n-1}$

$$= K\left(-\frac{dv_x}{dy}\right)^{n-1}$$

since $\frac{dv_x}{dy} < 0$ from sketch

$$\Rightarrow \tau_{yx} = K\left(-\frac{dv_x}{dy}\right)^{n-1}\frac{dv_x}{dy} = -K\left(-\frac{dv_x}{dy}\right)^n \quad (A)$$

Cauchy Momentum Equation:

x-comp: $\quad 0 = -\frac{\partial p}{\partial x} + \frac{d\tau_{yx}}{dy} + \rho g_x = -\frac{\partial p}{\partial x} + \frac{d\tau_{yx}}{dy} + \rho g\cos\alpha$

y-comp: $\quad 0 = -\frac{\partial p}{\partial y} + \rho g_y = -\frac{\partial p}{\partial y} + \rho g \sin\alpha$

$$\frac{\partial p}{\partial y} = \rho g\sin\alpha \Rightarrow p = (\rho g \sin\alpha)y + f(x) \qquad \frac{\partial p}{\partial x} = f'(x)$$

On free surface (at $y=0$) $p = p_0$ for all x so $\frac{\partial p}{\partial x} = 0$

$\Rightarrow \frac{\partial p}{\partial x} = 0$ throughout film. substituting into x-comp:

$$-\frac{d\tau_{yx}}{dy} = \rho g\cos\alpha \xrightarrow{\text{integr.}} \tau_{yx} = (-\rho g\cos\alpha)y + C_1$$

BC: At $y=0$ $\quad \tau_{yx}\big|_{\text{liquid}} = \tau_{yx}\big|_{\text{gas}} = 0$ $\quad$ stress is continuous across interface & air is inviscid

$$\Rightarrow C_1 = 0 \qquad \tau_{yx} = -\rho g\cos\alpha\, y$$

From Eqn A above:

$$\tau_{yx} = -K\left(-\frac{dv_x}{dy}\right)^n = -\rho g\cos\alpha\, y$$

$$-\frac{dv_x}{dy} = \left(\frac{\rho g\cos\alpha}{K}\right)^{\frac{1}{n}} y^{\frac{1}{n}} \xrightarrow{\text{integr.}} v_x = -\left(\frac{\rho g\cos\alpha}{K}\right)^{\frac{1}{n}}\int y^{\frac{1}{n}}\,dy$$

$$v_x = -\left(\frac{\rho g\cos\alpha}{K}\right)^{\frac{1}{n}}\left(\frac{n}{n+1}\right)y^{\frac{n+1}{n}} + C_2$$

Name Solution

BC: at $y=\delta$ $v_x = 0$ (no slip)

Applying BC: $C_2 = \left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{n+1}\right) \delta^{\frac{n+1}{n}}$

so $$\boxed{v_x = \left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{n+1}\right)\left[\delta^{\frac{n+1}{n}} - y^{\frac{n+1}{n}}\right] = \left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{n+1}\right)\delta^{\frac{n+1}{n}}\left[1-\left(\frac{y}{\delta}\right)^{\frac{n+1}{n}}\right]}$$

The thickness can be related to the volume rate of flow, which is

$$Q = W\int_0^\delta v_x \, dy \quad \Rightarrow \quad q = \frac{Q}{W} = \int_0^\delta v_x \, dy$$

$$q = \int_0^\delta \left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{n+1}\right) \delta^{\frac{n+1}{n}} \left[1 - \left(\frac{y}{\delta}\right)^{\frac{n+1}{n}}\right] dy$$

$$= \underbrace{\left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{n+1}\right) \delta^{\frac{n+1}{n}}}_{B} \left[y - \left(\frac{1}{\delta}\right)^{\frac{n+1}{n}} \left(\frac{n}{2n+1}\right) y^{\frac{2n+1}{n}} \right]_0^\delta$$

$$= B\left[\delta - \left(\frac{1}{\delta}\right)^{\frac{n+1}{n}} \left(\frac{n}{2n+1}\right)\delta^{\frac{2n+1}{n}}\right] = B\left[\delta - \left(\frac{n}{2n+1}\right)\delta\right] = B\delta\left(\frac{n+1}{2n+1}\right)$$

$$q = \left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{n+1}\right)\delta^{\frac{n+1}{n}} \delta \left(\frac{n+1}{2n+1}\right)$$

$$q = \left(\frac{\rho g \cos\alpha}{K}\right)^{\frac{1}{n}} \left(\frac{n}{2n+1}\right)\delta^{\frac{2n+1}{n}}$$

Rearranging: $$\boxed{\delta = \left[q\left(\frac{2n+1}{n}\right)\right]^{\frac{n}{2n+1}} \left(\frac{K}{\rho g \cos\alpha}\right)^{\frac{1}{2n+1}}}$$

b) $\eta = K\left|\frac{1}{2}\mathrm{II}\right|^{\left(\frac{n-1}{2}\right)}$ will be constant when $n=1$, giving $\eta = K$

in circled expressions, setting $n=1$ and $K=\eta$ gives:

Name Solution

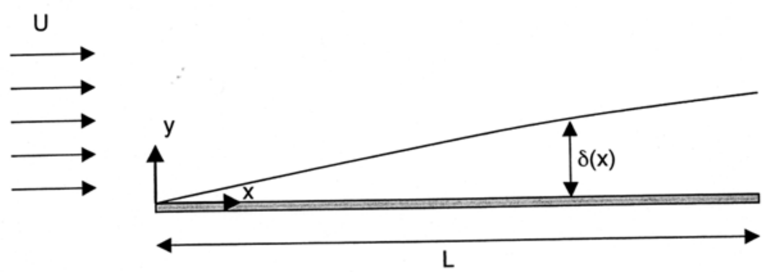
$$v_x = \left(\frac{\rho g \cos\alpha}{\eta} \right) \left(\frac{1}{2} \right) \delta^2 \left[1 - \left(\frac{y}{\delta} \right)^2 \right]$$

$$\delta = \left[3q \right]^{\frac{1}{3}} \left(\frac{\eta}{\rho g \cos\alpha} \right)^{\frac{1}{3}} = \left(\frac{3q\eta}{\rho g \cos\alpha} \right)^{\frac{1}{3}}$$

Name Solution

2. (20 pts) A fluid with uniform velocity U approaches a flat plate as shown in the figure below. The plate may be considered infinite in the z-direction and the flow is steady. A boundary layer develops, with a thickness δ that increases with distance down the plate. The characteristic length in the x-direction is L, the characteristic length in the y-direction is δ, and the characteristic velocity with which to scale v_x is U.

a) Use the continuity equation to show how v_y should be scaled with the quantities U, δ, and L.
b) Starting with the correct form of the x-component of the Navier-Stokes equation, determine the appropriate scaling (characteristic distance) for δ.



a) $v_z = 0$ since infinite in z-direction

Continuity is $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

Given $\tilde{v}_x = \frac{v_x}{U}$, $\tilde{x} = \frac{x}{L}$, and $\tilde{y} = \frac{y}{\delta}$. Let $\tilde{v}_y = \frac{v_y}{V}$ where V is to be determined.

sub into continuity:

$$\frac{U}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{V}{\delta}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$

or

$$\frac{U\delta}{LV}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$

in order for both terms to be non-negligible, need $\frac{U\delta}{LV} \sim 1$

$\Rightarrow$ $\boxed{V = \frac{U\delta}{L}}$ v_y should be scaled with $\frac{U\delta}{L}$.

b) From Table 7-4, the correct starting point is:

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

scaling pressure with Π: $\tilde{P} = \frac{P}{\Pi}$

Name Solution

scaling:

$$\rho\left(\frac{U^2}{L}\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}} + \frac{U^2\delta}{\delta L}\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right) = -\frac{\pi}{L}\frac{\partial\tilde{p}}{\partial\tilde{x}} + \eta\left(\frac{U}{L^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2} + \frac{U}{\delta^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}\right)$$

$$\rho\frac{U^2}{L}\left(\frac{\delta^2}{\eta U}\right)\Big(\underset{c}{\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}}} + \underset{d}{\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}}}\Big) = -\frac{\pi}{L}\left(\frac{\delta^2}{\eta U}\right)\frac{\partial\tilde{p}}{\partial\tilde{x}} + \underbrace{\Big(\underset{a}{\frac{\delta^2}{L^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2}} + \underset{b}{\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}}\Big)}$$

Viscous term a is much smaller than viscous term b since it is multiplied by $\frac{\delta^2}{L^2}$.

In boundary layer, want largest viscous term (b) to balance inertial terms (c or d).

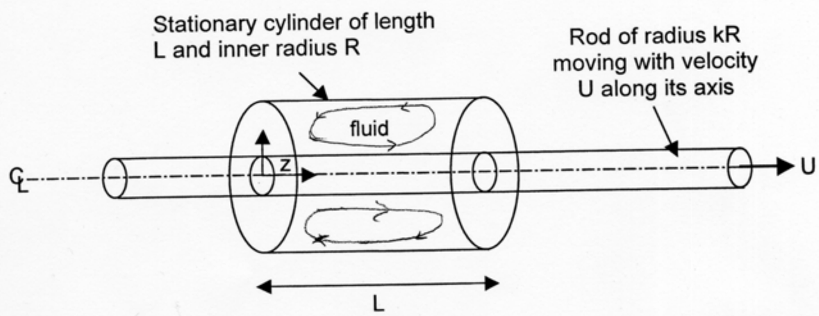
thus $\quad \frac{\rho U^2}{L}\frac{\delta^2}{\eta U} \simeq 1 \quad \Rightarrow \quad \delta^2 = \frac{\eta L}{\rho U}$

or $\quad \boxed{\delta \simeq \left(\frac{\eta L}{\rho U}\right)^{1/2}}$

Name Solution

3. (35 pts) A cylindrical rod of radius kR (where k < 1) moves with a constant velocity U through a stationary cylindrical container of radius R containing a Newtonian fluid. The rod is concentric with the cylinder and moves along its axis (as shown below). The ends of the cylinder are sealed so that the fluid remains in the container but circulates due to the motion of the rod. The length of the container L is comparable to R.

a) Show how you would solve for the velocity field in this problem. Do not attempt to solve the resultant equation(s), but clearly indicate assumptions, boundary conditions, and simplified equations.



Assume incompressible fluid
steady - state
axisymmetric so $v_r \neq v_r(\theta)$, $v_z \neq v_z(\theta)$, $v_\theta \neq v_\theta(\theta)$
assume $v_\theta = 0$

$$\Rightarrow \quad v_r = v_r(r, z) \quad , \quad v_z = v_z(r, z) \quad , \quad v_\theta = 0$$

Continuity : From table 7-1 :

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}$$

N-S : r-comp :

$$\rho\left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ- comp : $0 = 0$

z - comp :

$$\rho\left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Name Solution

Boundary conditions :

We need 2 BC's in r on v_r }
2 " " z on v_r } from r-comp of NS

2 BC's in r on v_z }
2 " " z on v_z } from z-comp of NS

These are :

No slip on rod and on sidewalls of cylinder

$v_r = 0, \quad v_z = U \quad$ at $\quad r = kR \quad$ for $\quad 0 < z < L$

$v_r = 0, \quad v_z = 0 \quad$ at $\quad r = R \quad$ for $\quad 0 < z < L$

No slip on container ends

$v_r = 0 \quad v_z = 0 \quad$ at $\quad z = 0 \quad$ for $\quad kR < r < R$

$v_r = 0 \quad v_z = 0 \quad$ at $\quad z = L \quad$ for $\quad kR < r < R$

Name Solution

b) How many boundary conditions are needed for the following partial differential equation, where H=H(p,T)? Indicate how many boundary conditions are needed for each independent variable. Also, indicate the total number of boundary conditions needed.

$$\frac{\partial^4 H}{\partial p^4} + T^2 \frac{\partial^3 H}{\partial T^3} + 5p\left(\frac{\partial H}{\partial p}\right)^6 = 1$$

4 boundary conditions in p

3 boundary conditions in T

$\Rightarrow$ 7 total

c) Plot the following function on the graph below: $yx^2 = 3$